On Some Approximations of the Sparse Super-Resolution Problem

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Joint work with V. Duval, M. Hockmann, S. Kunis, G. Peyré and M. Wageringel

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Sparse Super-Resolution

- **Problem:** Recover a signal from a few coarse linear measurements

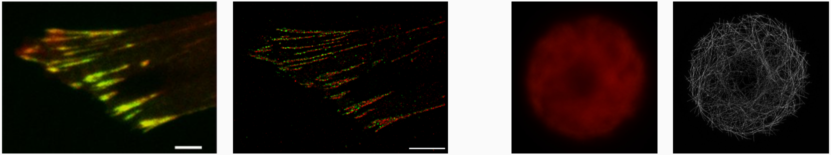
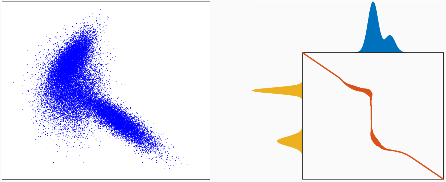


Image Processing: Fluorescence microcospy (source - `www.cellimagelibrary.org`), astronomical imaging, . . .



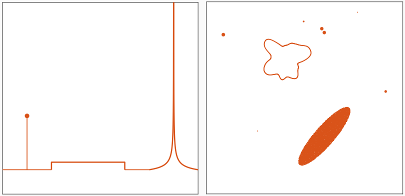
Machine Learning: Mixture estimation, optimal transport, . . .

- **Signals of interest are often structured:** pointwise sources, curves, graphs of functions, surfaces...

Data Model

- **Radon measures**
  $d \in \mathbb{N} \setminus \{0\}$, $\mathbb{T} \overset{\text{def.}}{=} \mathbb{R}/\mathbb{Z}$ (Torus),

$$\mu \in \mathcal{M}(\mathbb{T}^d)$$



Singular measures μ

- Topological dual of $\mathscr{C}(\mathbb{T}^d)$

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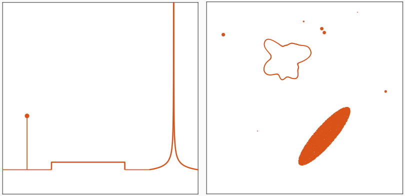
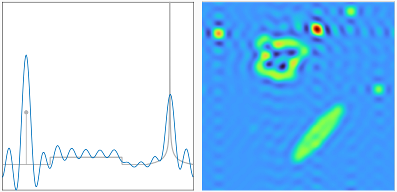


Singular measures μ

- Topological dual of $\mathscr{C}(\mathbb{T}^d)$

- **Trigonometric moments**
  $k \in \Omega \subset \mathbb{Z}^d$, here $\mathbb{Z}_n^d = \{-n, \ldots, n\}^d$

$$\hat{\mu}(k) \overset{\text{def.}}{=} \int_{\mathbb{T}^d} e^{-2\imath\pi \langle k, x \rangle} \mathrm{d}\mu(x)$$



Fourier partial sum $S_n\mu$ ($n = 13$)

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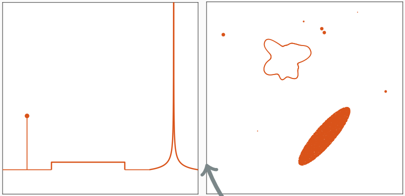

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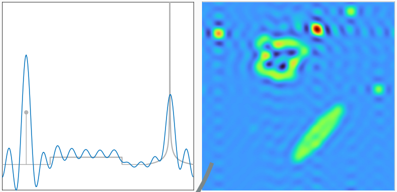
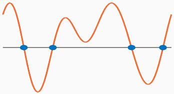


Fourier partial sum $S_n\mu$ ($n = 13$)

?

A Short Tour of Recovery Approaches



Prony

- Discrete
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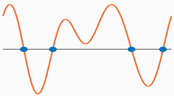

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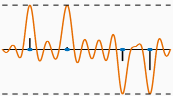

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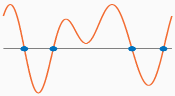

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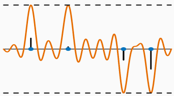

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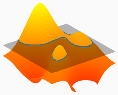

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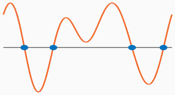

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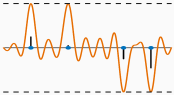

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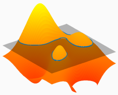

FRI

Christoffel

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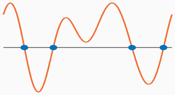
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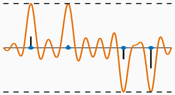
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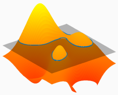
FRI


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- * In this presentation
  - empirically good generalization of Prony's method in the non-discrete case
  - polynomial approximations and interpolations, with rates in p-$\mathcal{W}$asserstein metric

Moment Matrix

For $\mu \in \mathcal{M}(\mathbb{T}^d)$ and $n \in \mathbb{N}$, we define the moment matrix of μ of order n by

$$T_n \stackrel{\text{def.}}{=} \Big(\hat{\mu}(k-l)\Big)_{k,l \in \mathbb{N}_n^d}$$

where $\mathbb{N}_n^d \stackrel{\text{def.}}{=} \left\{k \in \mathbb{N}^d \ ; \ \|k\|_\infty \leqslant n\right\}$.

Remark

- $T_n \in \mathbb{C}^{N \times N}$ with $N \stackrel{\text{def.}}{=} (n+1)^d$.
- T_n is multi-level Toeplitz: $T_{k+s,l} = T_{k,l-s}$, for all $k, s, l \in \mathbb{Z}^d$
- for instance with $d = 1$

$$T_n = \begin{bmatrix} \hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \dots \\ \hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(1) & \dots \\ \hat{\mu}(2) & \hat{\mu}(1) & \hat{\mu}(0) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Overview

1. **An Extension of Prony's Method**
2. **Polynomial Approximations**
3. **Polynomial Interpolation**
4. **Conclusion**

An Extension of Prony's Method

Discrete Recovery

Algorithm 1: Multivariate recovery for flat data

Input: T_n SDP, Toeplitz, flat matrix

Output: $x_1, \ldots, x_r \in \mathbb{T}^d$

for $i = 1$ **to** d **do**

  Compute shifted matrix $T_{n-1}^{(i)}$

  Compute svd $T_{n-1} = U \Sigma U^*$

  Compute multiplication matrices $X_i = \Sigma^{-1} U^* T_{n-1}^{(i)} U$

end

Compute joint diagonalization basis P

  *Diagonalize $X_\alpha = \sum \alpha_i X_i$, for random $\alpha_i \in [0, 1]$

Return $x_{j,i} = -\frac{1}{2\pi} \arg\left(P^{-1} X_i P\right)_{jj}, \quad j = 1, \ldots, r, \quad i = 1, \ldots, d$

* **Lemma.** If the X_is are jointly diagonalizable, then with probability one X_α is non-derogatory (*i.e.* all eigenspaces are of dimension 1), with eigenvalues

$$\nu_j = \sum_{i=1}^{d} \alpha_i e^{2\imath\pi x_{j,i}}, \quad j = 1, \ldots, s.$$

Non-Discrete Recovery

- If μ is not discrete, we essentially lose the flatness of T_n
- Guarantees of robustness in the non-flat case exist [Klep, Povh, and Volčič, 2018]
- What is the numerical perspective?

Algorithm 2: Multivariate recovery for ~~flat~~ data

Input: T_n SDP, Toeplitz, ~~flat~~ matrix

Output: $x_1, \ldots, x_r \in \mathbb{T}^d$

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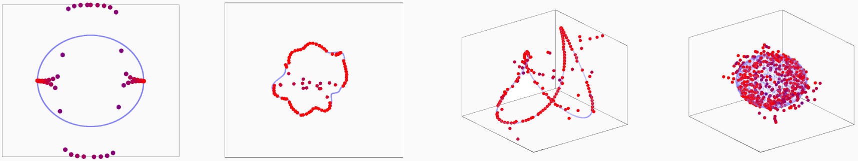


Diagonality Criterion

- X_i non-commuting, not jointly diagonalizable

  $\rightarrow$ find a basis in which they are "almost" diagonal
- Off-diagonal criterion to minimize

$$\mathcal{O}(P) \stackrel{\text{def.}}{=} \sum_i \sum_{\alpha \neq \beta} (P X_i P^{-1})^2_{\alpha\beta}$$

  - criterion used *e.g.* in [Cardoso and Souloumiac, 1996],[Joho and Rahbar, 2002] for blind source separation, but restricted to orthogonal matrices
  - X_i are not Hermitian
  - Riemannian optimization over $\mathrm{GL}_r(\mathbb{C})$

Quasi-Newton updates

- Invertibility is maintained using updates of the form $P_{t+1} = (I_r + \mathcal{E})P_t$
- Taylor expansion: $\mathcal{O}((I+\mathcal{E})P) = \mathcal{O}(T) + \langle G(P), \mathcal{E}\rangle + \langle H(P)\mathcal{E}, \mathcal{E}\rangle + o(\|\mathcal{E}\|^2)$
  - Relative gradient: with $\underline{Y} = Y - \mathrm{Diag}(Y)$ and $Y_i = PX_iP^{-1}$

$$G(P) = \sum_i \underline{Y}_i Y_i^* - Y_i^* \underline{Y}_i$$

  - Relative Hessian: use diagonal approximation [Ablin, Cardoso, and Gramfort, 2019]. When Y_i are diagonal,

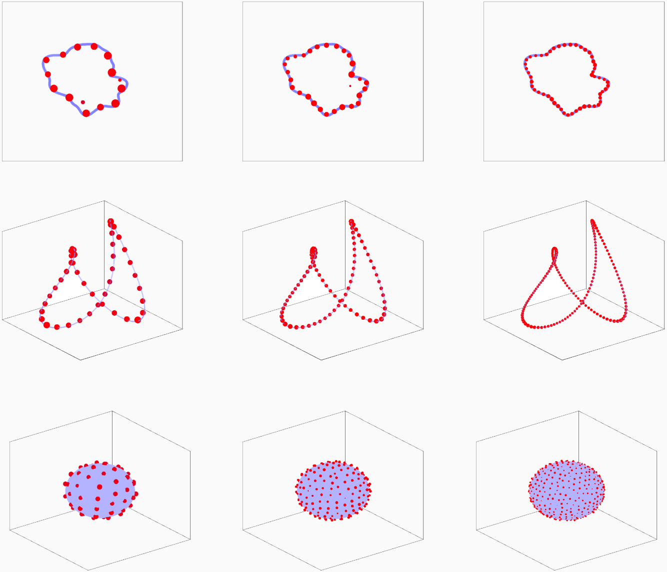
$$\tilde{H}_{pqrs}(P) = \delta_{pr}\delta_{qs} \sum_i |(Y_i)_{pp} - (Y_i)_{qq}|^2$$

$\rightarrow$ $\tilde{H}$ is sparse and positive semidefinite

- Quasi-Newton update: $P_{t+1} = (I + \alpha\mathcal{E}_t)P_t$, where α is found by linesearch and

$$\mathcal{E}_t = -(\tilde{H}(P_t) + \beta I)^{-1} \cdot G(P_t)$$

Results



$n = 5$ $n = 10$ $n = 20$ (15 for sphere)

Polynomial Approximations

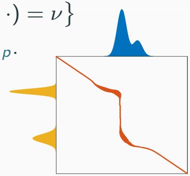
Wasserstein distances

- **Distances between measures:** f-divergences, discrepancies, Wasserstein distances, ...

$$\mathcal{W}_p^p(\mu,\nu) = \inf \left\{ \int d(x,y)^p \mathrm{d}\pi(x,y) \; ; \; \pi \in \Pi(\mu,\nu) \right\}$$

[Kantorovich, 1942]

  - set of couplings: $\Pi(\mu,\nu) \stackrel{\text{def.}}{=} \{\pi \in \mathcal{M}_+(\mathbb{T}^d \times \mathbb{T}^d) \; ; \; \pi(\cdot,\mathbb{T}^d) = \mu, \; \pi(\mathbb{T}^d,\cdot) = \nu\}$
  - d distance on $\mathbb{T}^d$: we use $d(x,y) = \|x-y\|_{p,\mathbb{T}} \stackrel{\text{def.}}{=} \min_{k\in\mathbb{Z}^d} \|x-y+k\|_p$.



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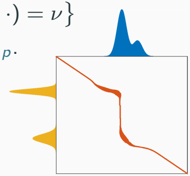

- **Dual problem:** $\mathcal{W}_1$ further admits the practical dual formulation

$$\mathcal{W}_1(\mu,\nu) = \sup\left\{\int f \mathrm{d}(\mu-\nu)\ ;\ f \in \mathrm{Lip}_1\right\}$$

  - requires only $\mu(\mathbb{T}^d) = \nu(\mathbb{T}^d)$ (no positivity)
  - $\mathrm{Lip}_1 \overset{\text{def.}}{=} \{f \in \mathscr{C}(\mathbb{T}^d)\ ;\ |f(x)-f(y)| \leqslant \|x-y\|_{1,\mathbb{T}},\ \forall x,y \in \mathbb{T}^d\}$

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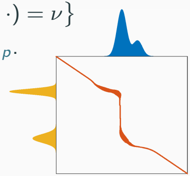

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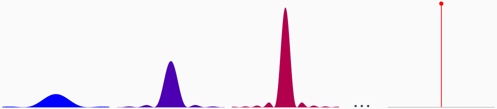
- $\mathcal{W}_p$ metrizes the weak* topology (on compact sets) [Santambrogio, 2015]

$$\left(\forall \varphi \in \mathscr{C}(\mathbb{T}^d),\ \int \varphi \mathrm{d}\mu_n \to \int \varphi \mathrm{d}\mu\right) \iff \mathcal{W}_p(\mu_n,\mu) \to 0$$



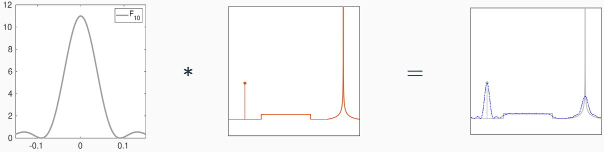
Fejér approximation

- The Fejér kernel F_n is defined by

$$F_n(x) \overset{\text{def.}}{=} \frac{1}{N} \prod_{i=1}^{d} \frac{\sin^2\left((n+1)\pi x_i\right)}{\sin^2\left(\pi x_i\right)}$$

- For arbitrary μ, consider the polynomial

$$p_n \overset{\text{def.}}{=} F_n * \mu, \quad i.e. \quad p_n(x) = \int F_n(y-x)\mathrm{d}\mu(y)$$



- p_n has a simple expression in terms of the moment matrix:

$$p_n(x) = N^{-1} v_n(x)^* T_n v_n(x), \quad \text{where} \quad v_n(x) = \left(e^{2\imath\pi\langle k,\, x\rangle}\right)_{k\in\mathbb{N}_n^d}$$

- p_n can be computed using Fast Fourier Transforms:

$$p_n\left(\frac{j}{M}\right) = N^{-1} \sum_{k\in\mathbb{Z}_n^d} w(k)\hat{\mu}(k)e^{2\imath\pi\langle \frac{k}{M},\, j\rangle}, \quad \text{where} \quad w(k) = \widehat{F_n}(k) = \prod_{i=1}^{d}(1 - \frac{|k_i|}{n+1})$$

Convergence (Fejér)

Theorem (Weak-* convergence). Assuming (only) that μ has finite total variation, we have that $p_n \rightharpoonup \mu$. More precisely,

$$\begin{cases} \mathcal{W}_1(p_n, \mu) \leqslant \frac{d}{\pi^2} \frac{\log(n+1) + 3}{n} \\ \mathcal{W}_p(p_n, \mu) \leqslant \left(\frac{2d}{p-1}\right)^{1/p} \frac{1}{(n+1)^{1/p}}, \quad p > 1 \end{cases}$$

These bounds are tight in the worst case:

$$\begin{cases} \frac{d}{\pi^2}\left(\frac{\log(n+2)}{n+1} + \frac{1}{n+3}\right) \leqslant \sup_{\mu \in \mathcal{M}(\mathbb{T}^d)} \mathcal{W}_1(p_n, \mu) \\ \left(\frac{d}{2\pi^2(p-1)}\right)^{1/p} \frac{1}{4n^{1/p}} \leqslant \sup_{\mu \in \mathcal{M}(\mathbb{T}^d)} \mathcal{W}_p(p_n, \mu), \quad p > 1 \end{cases}$$

First step of the proof:

Use the dual formulation of $\mathcal{W}_p$ to derive the relation

$$\mathcal{W}_p(F_n * \mu, \mu)^p \leqslant \int F_n(x) \|x\|_p^p \mathrm{d}x.$$

Saturation

- Further assumptions on μ do not improve so much this bound.

Theorem (Saturation). For every measure $\mu \in \mathcal{M}(\mathbb{T}^d)$ not being the Lebesgue measure, there exists a constant c such that

$$\mathcal{W}_1(p_n, \mu) \geqslant \frac{c}{n+1}$$

- For instance $\mathrm{d}\mu = (1 + \cos(2\pi x))\mathrm{d}x =: w(x)\mathrm{d}x$ yields

$$\mathcal{W}_1(p_n, w) \geqslant \frac{1}{4\pi(n+1)}$$

Powers of Fejér approximations

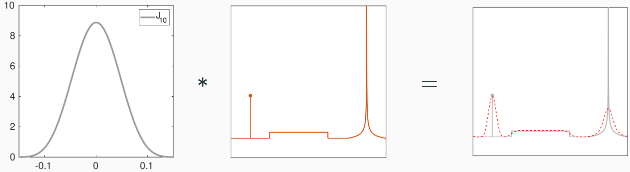
- We define the higher localized kernels

$$r = \left\lceil \frac{p}{2} + 1 \right\rceil, \quad m = \left\lfloor \frac{n}{r} \right\rfloor, \quad K_{n,p}(x) \overset{\text{def.}}{=} C_{m,d} \prod_{i=1}^{d} \frac{\sin^{2r}((m+1)\pi x_i)}{\sin^{2r}(\pi x_i)},$$

→ For instance, with $p = 1$, $K_{n,1}(x) = \frac{3}{m(2m^2+1)} \frac{\sin^4((m+1)\pi x}{\sin^4(\pi x)}$ is the Jackson kernel

- Consider the polynomial

$$q_{n,p} \overset{\text{def.}}{=} K_{n,p} * \mu, \quad \textit{i.e.} \quad q_{n,p}(x) = \int K_{n,p}(x-y)\mathrm{d}\mu(y)$$



- $q_{n,p}$ is of degree at most n
- $q_{n,p}$ can be computed with Fast Fourier Transforms

$$q_{n,p}\left(\frac{j}{M}\right) = C_n^{-1} \sum_k \widehat{K_{n,p}}(k)\hat{\mu}(k) e^{2\imath\pi\langle \frac{k}{M}, j\rangle}$$

Convergence

Theorem. (Weak* convergence) Assuming that μ has finite total variation, we have that $q_{n,p} \rightharpoonup \mu$. More precisely, there exists C_p independent of n such that

$$\mathcal{W}_p(q_{n,p}, \mu) \leqslant \frac{C_p}{n}.$$

This rate is sharp in the worst-case

$$\frac{d^{(1-p)/p}}{4(n+1)} \leqslant \sup_{\mu \in \mathcal{M}(\mathbb{T}^d)} \mathcal{W}_p(q_{n,p}, \mu)$$

- In particular, with the Jackson kernel, we have for instance

$$\mathcal{W}_1(K_{n,1} * \mu, \mu) \leqslant \frac{3d}{2(n+2)}$$

Best Polynomial Approximation

Theorem (Worst-case bound). For every $d, n \in \mathbb{N}$, for every $\mu \in \mathcal{M}(\mathbb{T}^d)$ with finite total variation, there exists a polynomial of best approximation in the Wasserstein-1 distance. Moreover

$$\sup_{\mu \in \mathcal{M}(\mathbb{T}^d)} \min_{\deg(p) \leqslant n} \mathcal{W}_p(p, \mu) \geqslant \frac{d^{(1-p)/p}}{4(n+1)}.$$

Sketch of proof:

- Best approximation in the worst-case:

$$\begin{aligned} \sup_{\mu} \min_{p} \mathcal{W}_1(p, \mu) &\geqslant \min_{p} \mathcal{W}_1(p, \delta_0) \\ &= \min_{p} \sup_{\mathrm{Lip}(f) \leqslant 1} \|f - \check{p} * f\|_\infty \qquad (\check{p}(x) = p(-x)) \\ &\geqslant \sup_{\mathrm{Lip}(f) \leqslant 1} \min_{p} \|f - p\|_\infty \end{aligned}$$

  → worst-case error for best polynomial approximation of Lipschitz functions
  → + generalization of a univariate argument of [Fisher, 1977] to the multivariate case

- Extend to $\mathcal{W}_p$ using $\mathcal{W}_p(\mu, \nu) \geqslant d^{/1-p)/p} \mathcal{W}_1(\mu, \nu)$ from Jensen's and Hölder's inequality.

Sharpness

- For this worst-case bound, sharpness is revealed in the univariate case

Theorem. With $x \in \mathbb{T}$ we have $\mathcal{W}_1(p_\star, \delta_x) = \frac{1}{4}(n+1)^{-1}$.

- Proof involves the relation

$$\mathcal{W}_1(\mu, \nu) = \|\mathcal{B}_1 * \mu - \mathcal{B}_1 * \nu\|_{\mathrm{L}^1}, \quad \text{where} \quad \mathcal{B}_1 : t \in \mathbb{T} \mapsto \frac{1}{2} - t$$

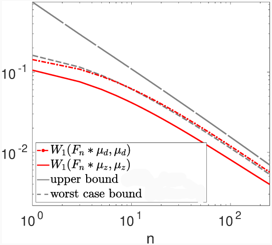
(Periodic analog of the cumulative distribution formulation of $\mathcal{W}_1$ on $\mathbb{R}$)

- Transfer (by deconvolution) results on unicity of best L^1-approximation to unicity of our best polynomial approximation in some cases (*e.g.* μ a.c., or $\mu = \delta_x$)

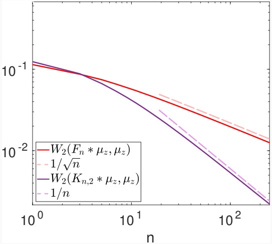
- De La Vallée Poussin approximation [Mhaskar, 2019] also achieves optimal rate but dependency with d is worse in the constant

Numerical Verification

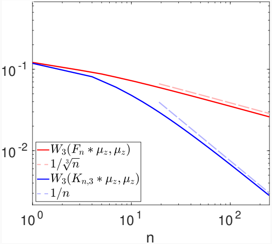
- **Test on two measures:**
  - a discrete measure μ_d, $s = 15$
  - a (discretized) algebraic curve μ_z, $s = 3000$
- Semidiscrete algorithm to compute the Wasserstein distance between polynomial density and singular measure.



(x) $\mathcal{W}_1$-rates for Fejér (μ_d, μ_z)



(y) W_2-rates for Fejér and $K_{n,1}$ (μ_z)



(z) W_3-rates for Fejér and $K_{n,2}$ (μ_z)

Polynomial Interpolation

Interpolating Polynomial

- The singular value decomposition: $T_n = \sum_{j=1}^{r} \sigma_j u_j^{(n)} v_j^{(n)*}$ allows to define

$$p_{1,n}(x) = \frac{1}{N} \sum_{j=1}^{r} |u_j^{(n)}(x)|^2$$

$\rightarrow$ unweighted counterpart of $p_n = N^{-1} e(x)^* T_n e(x) = N^{-1} \sum \sigma_j u_j^{(n)}(x) v_j^{(n)}(x)^*$.
Note that $0 \leqslant p_{1,n} \leqslant 1$.

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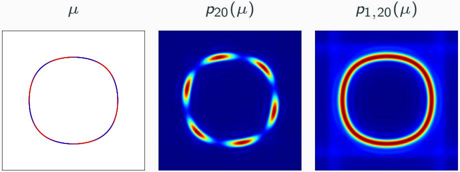
Note that $0 \leqslant p_{1,n} \leqslant 1$.

- Assume that $Z \stackrel{\text{def.}}{=} \operatorname{Supp} \mu$ is an algebraic variety (*i.e.* defined by polynomial equations) Let $\mathcal{V}(\operatorname{Ker} T_n)$ be the set of common roots of all polynomials in $\operatorname{Ker} T_n$.

Theorem (Interpolation). If $\mathcal{V}(\operatorname{Ker} T_n) = Z$, then $p_{1,n}(x) = 1$ iff $x \in V$.

→ $\mathcal{V}(\operatorname{Ker} T_n) = Z$ always holds for sufficiently large n if μ is discrete [Kunis et al., 2016],[Sauer, 2017] or nonnegative [Wageringel, 2022]



Pointwise convergence

- We assume that $Z \neq \mathbb{T}^d$

Theorem. Let $y \in \mathbb{T}^d \setminus Z$, and let g be a polynomial of max-degree m such that $g(y) \neq 0$ and g vanishes on Z. Then, for all $n \geqslant m$,

$$p_{1,n+m}(y) \leqslant \frac{\|g\|_{L^2}^2}{|g(y)|} \frac{m(4m+2)^d}{n+1} + \frac{dm}{n+m+1}$$

- In combination with the interpolation property, this proves pointwise convergence to the characteristic function of the support, with rate $O(n^{-1})$.

The Discrete Case

- If $\mu = \sum_{j=1}^{r} \lambda_j \delta_{x_j}$, stronger results are derived with the help of the Vandermonde decomposition of T_n

Theorem (Pointwise convergence). Let $x \neq x_j$ for all j. If $n+1 > \frac{4d}{\min_{j \neq l} \|x_j - x_l\|_\infty}$, then

$$p_{1,n}(x) \leqslant \frac{1}{3(n+1)^2} \frac{\lambda_{\max}}{\lambda_{\min}} \sum \frac{1}{\|x - x_j\|_\infty^2}$$

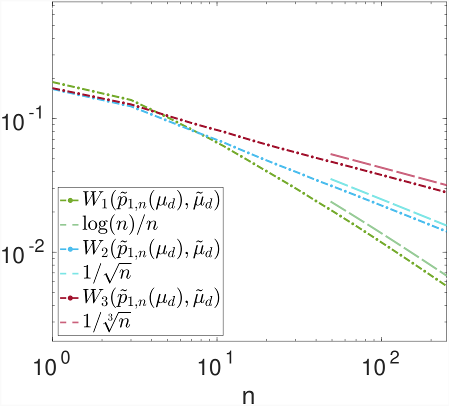
Theorem (Weak* convergence). Let $\tilde{p}_{1,n} \stackrel{\text{def.}}{=} p_{1,n}/\|p_{1,n}\|_{L^1}$. We have

$$\tilde{p}_{1,n} \rightharpoonup \tilde{\mu} \stackrel{\text{def.}}{=} \frac{1}{r} \sum_{j=1}^{r} \delta_{x_j}.$$

More specifically

$$\begin{cases} \mathcal{W}_1(\tilde{p}_{1,n}, \tilde{\mu}) = O(\frac{\log n}{n}) \\ \mathcal{W}_p(\tilde{p}_{1,n}, \tilde{\mu}) = O(n^{-1/p}) \end{cases}$$

Numerical Verification



$\mathcal{W}_{\{1,2,3\}}$-rates for $\tilde{p}_{1,n}$

Conclusion

Conclusion

Summary.

- Two "dual" approaches to the recovery of non-discrete measures from moments
- Dedicated solver for diagonalization is key in Prony's method
- New insights on Wasserstein approximation of measures/support
- Computationally efficient polynomial approximations

Outlook.

- Extension to the noisy regime
- Connection with Christoffel functions

One preprint available: arXiv.2203.10531

Thank you for your attention!

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