Paul Catala Wintersemester 2023/2024

Fallstudien der Mathematischen Modellbildung: Teil 2 (MA 2902)

Themen für Hausarbeiten

Thema 1 (MEG/EEG imaging)

- Discuss the general principle of MEG/EEG imaging [1]. Describe the corresponding linear model(s), and explain why the associated inverse problem is ill-posed.
- Discuss the following priors on the sources, presented in [1]: Sp1, Sp2, Sp3, Sp4, Te1, Te2. In particular, give examples of possible regularizer functionals.
- Which regularizer could be used to favor sources with low energy and a sparse spatial distribution (at a fixed time sample t)? Describe an algorithm to solve the corresponding regularized least-squares problem.
- Define the ℓ^1/ℓ^2-norm described *e.g.* in the introduction of [4]. Comment on its use as a regularizer for the MEG/EEG recovery problem.

Thema 2 (Total least-squares)

- Define and discuss the total least-squares problem, which is described *e.g.* in [3]. Comment in particular on the differences with the ordinary least-squares problem.
- Here we use the notations of [3]. Let $C \overset{\text{def.}}{=} \begin{bmatrix} A & B \end{bmatrix}$ and $\delta C \overset{\text{def.}}{=} \begin{bmatrix} \delta A & \delta B \end{bmatrix}$. Formulate the constraint in (TLS1) in terms of C and δC. Deduce a lower bound on $\|\delta C\|_F$, and a minimizer δC_*, using Eckart-Young theorem (see *e.g.* Lemma 4). Show that
$$\operatorname{Ker}(C + \delta C_*) = \operatorname{Ran} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}.$$
Deduce a proof of Theorem 2.
- It is said in [3] that for a vector right-hand side b, the solution x_* of (TLS1) satisfies
$$(A^\top A - \sigma_{n+1}^2 I)x_* = A^\top b, \tag{1}$$
where σ_{n+1} is the last singular value of C. Compare with the normal equations for an ordinary least-squares problem. What can you deduce about the conditioning of TLS compared to LS? Give the Lagrangian of the (TLS1) problem. What could be a strategy to derive (1)?
- Explain the inverse problem tackled in [2], and why TLS may be useful in this case.

References

[1] H. Becker, L. Albera, P. Comon, R. Gribonval, F. Wendling, and I. Merlet. Brain-source imaging: from sparse to tensor models. *IEEE Signal Proc. Mag.*, 32(6):100–112, 2015.

[2] N. Bose, H. Kim, and H. Valenzuela. Recursive total least-squares algorithm for image reconstruction from noisy, undersampled frames. *Mult. Sys. Sig. Proc.*, 4:253–268, 1993.

[3] I. Markovsky and S. Van Huffel. Overview of total least-squares methods. *Sig. Proc.*, 87(10):2283–2302, 2007.

[4] W. Ou, P. Golland, and M. Hämäläinen. A distributed spatio-temporal EEG/MEG inverse solver. *Med. Ima. Comput. Comput. Assist. Interv. (MICCAI)*, 11(1):26–34, 2008.