Rank Regularization

Fallstudien der mathematischen Modellbildung, Teil 2

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Matrix Completion

product\user	1	2	3	4	...
1		3		1	
2	6		5	1	
3	3	2			
⋮			1	5	

- Recovery approach: every user's rating is a linear combination of a few representative ones

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- With $y = AX_0$, A being a restriction operator, solve

$$\textbf{min} \quad \textbf{rank}(X) \quad \text{s.t.} \quad A(X) = A(X_0)$$

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- $\mathbf{rank}\, X = \#\{\text{non-zero singular values}\} = \|\sigma\|_0$ where $X = U\, \mathbf{diag}(\sigma) V^\top$
  $\Rightarrow$ Non-convex, combinatorial problem: consider convex relaxation

- You already know operator norms

$$\|X\|_{E\to F} = \sup_{\|v\|_E \leqslant 1} \|Xv\|_F$$

In particular, if $\|\cdot\|_E = \|\cdot\|_F = \|\cdot\|_2$ then $\|X\|_{2\to 2} = \sigma_{\max}(X)$, also called spectral norm.

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- You already know $L_{p,q}$-norms.

$$\|X\|_{p,q} = \left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}|X_{ij}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

In particular, $\|X\|_{2,2} = \sqrt{\sum X_{ij}^2} = \sqrt{\mathrm{Tr}(X^\top X)} = \sqrt{\sum_{i=1}^{\min(m,n)} \sigma_i(X)^2}$ is the Frobenius norm.

Definition (Trace inner product).

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- p-Schatten norms

$$\|X\|_p = \left(\sum_{i=1}^{\min(m,n)} \sigma_i(X)^p \right)^{\frac{1}{p}}$$

In particular,

- $\|X\|_2 = \|X\|_F$
- $\|X\|_\infty = \|X\|_{2\to 2}$ (often denoted $\|X\|_2$, *e.g.* in MATLAB...)
- $\|X\|_1$: nuclear norm. Also denoted $\|X\|_*$

- $\|\cdot\|_*$ is the dual norm of $\|\cdot\|_\infty$
  Recall the dual norm is: $\|y\|_* := \sup\{\langle x, y\rangle \; ; \; \|x\| \leqslant 1\}$

Theorem (Von Neumann's trace inequality). For $X, Y \in \mathbb{R}^{m\times n}$,

$$|\langle X, Y\rangle| \leqslant \sum_i \sigma_i(X)\sigma_i(Y)$$

with equality if X and Y share the same singular vectors.

Hence, assuming that $\|X\|_\infty = \sigma_{\max}(X) \leqslant 1$,

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- $\|X\|_*$ is the convex envelope of the rank

Theorem (Nuclear norm and rank). The convex envelope of $X \mapsto \mathrm{rank}(X)$ over the set $\|X\|_\infty \leqslant 1$, is $\|X\|_*$.

Subdifferential of the Nuclear Norm

- Recall, for a vector $\sigma = (\sigma_1, \dots, \sigma_r, 0, \dots, 0)$,

$$\partial \|\sigma\|_1 = \{(1, \dots, 1, x_{r+1}, \dots, x_n)\ ;\ |x_i| \leqslant 1\}$$

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- ***Lemma (Watson, 1992).*** Let $\Phi(X) = \Phi(\Sigma)$ when $X = U\Sigma V^\top$, $\Sigma = \mathrm{Diag}(\sigma)$. Then

$$\partial\Phi(X) = \mathrm{conv}\left\{U\,\mathrm{diag}(d)V^\top\ ;\ d \in \partial\Phi(\sigma)\right\}$$

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Here $\Phi(X) = \|X\|_*$, and for any $d \in \partial\|\sigma\|_1$,

$$U\,\mathrm{diag}(d)V^\top = U_1V_1^\top + U_2\,\mathrm{diag}(w)V_2^\top$$

where $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$, where $|w_i| \leqslant 1$ for all i.

U_1, V_1 are fixed, U_2, V_2 are arbitrary orthogonal complements: if $U^{(2)}, V^{(2)}$ are bases of $(\mathrm{Ran}\,U_1)^\perp$, $(\mathrm{Ran}\,V_1)^\perp$ respectively, then

$$U_2 = U^{(2)}Y, \quad \text{and} \quad V_2 = V^{(2)}Z$$

for some *orthogonal* $Y \in \mathbb{R}^{(m-r)\times(m-r)}, Z \in \mathbb{R}^{(n-r)\times(n-r)}$.

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Convex envelope: let $\sum \lambda_i = 1$, $\lambda_i \geqslant 0$, an element in the convex hull is of the form

$$U_1V_1^\top + \sum_i \lambda_i U_{2,i}\,\mathbf{diag}(w_i)V_{2,i}^\top = U_1V_1^\top + U^{(2)}\left(\sum_i \lambda_i Y_i\,\mathbf{diag}(w_i)Z_i^\top\right)V^{(2)\top}$$

and $\|\sum \lambda_i Y_i\,\mathbf{diag}(w_i)Z_i^\top\|_\infty \leqslant \sum \lambda_i \max(|w_i|) \leqslant 1$

- Hence the convex envelope is

$$\partial\|X\|_1 = \left\{U_1 V_1^\top + U^{(2)} W V^{(2)\top} \; ; \; \|W\|_\infty \leqslant 1\right\}$$

or equivalently

$$\partial\|X\|_1 = \left\{U_1 V_1^\top + Z \; ; \; U_1^\top Z = 0, \quad Z V_1 = 0, \quad \|Z\|_\infty \leqslant 1\right\}$$

Proximal Operator of the Nuclear Norm

- *Proposition (Proximity operator of the nuclear norm).*

$$\text{argmin}_X \, \gamma\|X\|_* + \frac{1}{2}\|X - Y\|_F^2 = US_\gamma(\boldsymbol{\Sigma})V^T =: S_\gamma(Y)$$

where $Y = U\boldsymbol{\Sigma}V^*$ and S_λ is the soft-thresholding operator, applied entry-wise to the diagonal matrix of singular values ($S_\lambda(\sigma) = (\sigma - \lambda)_+$, since $\sigma \geqslant 0$).

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Proof. The objective is strongly convex, so X is the (unique) minimizer if and only if

$$0 \in \partial\|X\|_* + \frac{1}{\gamma}(X - Y)$$

Let $Y = U_1\mathbf{\Sigma}_1V_1^\top + U_2\mathbf{\Sigma}_2V_2^\top$, where $\mathbf{diag}(\mathbf{\Sigma}_1) > \gamma$ and $\mathbf{diag}(\mathbf{\Sigma}_2) \leqslant \gamma$, and let

$$\hat{X} = U_1(\mathbf{\Sigma}_1 - \gamma)V_1^\top = US_\gamma(\mathbf{\Sigma})V^\top.$$

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Then

$$Y - \hat{X} = \gamma(U_1V_1^\top + U_2\,\mathbf{Diag}(d)V_2^\top) \quad \text{where} \quad |d_i| \leqslant 1,$$

and therefore

$$\frac{1}{\gamma}(Y - \hat{X}) \in \partial\|\hat{X}\|_*$$

i.e.

$$0 \in \partial\|\hat{X}\|_* + \frac{1}{\gamma}(\hat{X} - Y)$$

□

Nuclear Norm Minimization

- Relax rank minimization problem into

$$\min \|X\|_* \quad \text{s.t.} \quad A(X) = A(X_0), \tag{NN}$$

or in the penalized form

$$\min \|X\|_* + \frac{\lambda}{2}\|A(X - X_0)\|_F^2$$

- Forward-Backward splitting scheme:
  - gradient descent step: $Y_k = X_k + \delta A^\top A(X - X_0) = X_k - \delta A(X - X_0)$
  - proximal step: $X_{k+1} = S_\lambda(Y_k)$

- When $X \succeq 0$ (semidefinite positive), then $\|X\|_* = \mathrm{Tr}(X) = \sum \sigma_i$, and the minimization problem becomes

$$\min \mathrm{Tr}(X) \quad \text{s.t.} \quad A(X) = y, \quad X \succeq 0$$

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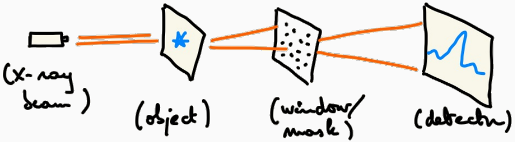
- This type of problem is called a semidefinite programming problem. This is a well-known type of problem, with dedicated solvers (*e.g.* interior points algorithms)
- There is actually a strong connection between (NN) and semidefinite programming: one can show that (NN) is equivalent to

$$\min \mathbf{Tr}(Y) \quad \text{s.t.} \quad Y = \begin{bmatrix} W_1 & X \\ X^\top & W_2 \end{bmatrix} \succeq 0, \quad A(X) = y$$

for some W_1,W_2.

- Diffraction imaging:



Intensity measurements are of the form

$$y_{ik} = |(FM_k x)_i|^2, \quad i \in 2D\text{-grid}$$

where F is the Fourier transform, M_k is a mask. More simply, $y_j = |a_j^\top x|^2$ for some a_j, $j = 1, \ldots, m > n$. The inverse problem of recovering x from y is called phase retrieval because the modulus "kills" the phase.

- Phase retrieval is a feasibility problem

$$\text{find} \quad x \quad \text{s.t.} \quad y_j = |a_j^\top x|^2$$

which is difficult (non-convex).

- Idea: look for $X = xx^\top$, with

$$y_j = |a_j^\top x|^2 = x^\top a_j a_j^\top x = \mathbf{Tr}(a_j a_j^\top xx^\top) = \mathbf{Tr}(A_j X) = \langle A_j, X\rangle$$

and $\mathbf{rank}(X) = 1, X \succeq 0$. So the problem can "lifted" as

$$\min \mathbf{rank}(X) \quad \text{s.t.} \quad X \succeq 0, \quad \langle A_j, X\rangle = y_j \ \forall j$$

- Relax into a convex SDP as

$$\min \mathbf{Tr}(X) \quad \text{s.t.} \quad X \succeq 0, \quad \langle A_j, X\rangle = y_j$$

[1] *PhaseLift: Exact and Stable Signal Recovery from Magnitude Measurements vie Convex Programming, Candès et al., 2013*