RECAP

- $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ finite - dimensional Hilbert spaces
- **Def** : The operator norm of $A \in \mathscr{L}(E,F)$ is $\|A\|_{E,F} := \sup_{x \neq 0} \frac{\|Ax\|_F}{\|x\|_E}$ $\Longrightarrow$ $\left[\begin{array}{l} \forall x \in E , \\ \|Ax\|_F \leq \|A\|_{E,F} \|x\|_E \end{array}\right.$

RECAP

$Ax = y$

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$$\Longrightarrow \left[\begin{array}{l} \forall x \in E \, , \\ \|Ax\|_F \leq \|A\|_{E,F} \, \|x\|_E \end{array} \right.$$

- **Def** : The condition number of $A \in \mathcal{L}(E,F)$ is $\kappa(A) = \|A\|_{E,F} \, \|A^{-1}\|_{F,E}$
- We set $\|\cdot\|_E = \|\cdot\|_F = \|\cdot\|_2$. In that case, $\kappa_2(A)$ may be explicitly given using the SVD of A.
- **Thm** : let $A \in \mathbb{R}^{m \times n}$ s.t. rank $A = r$. Then there exist $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ orthogonal, and $\Sigma \in \mathbb{R}^{m \times n}$ s.t.

$$A = U \Sigma V^T \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ 0 & & & 0 \end{pmatrix}$$

and $\sigma_1 \geq \dots \geq \sigma_r > 0$

$$U^T U = U U^T = I$$

Proof

$\in \mathbb{R}^{n\times n}$

1) The matrix A^TA is symmetric, hence diagonalisable in orthonormal basis of $\mathbb{R}^n$: $\begin{cases} v_1, \dots, v_n & \text{eigenvectors} \\ \lambda_1, \dots, \lambda_n & \text{eigenvalues} \end{cases}$, $A^TA = V \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} V^T$

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2) The matrix A^TA is semidefinite positive : $\forall x \in \mathbb{R}^n$, $x^TA^TAx = \|Ax\|^2 \geq 0$

hence $\lambda_i \geq 0$, $i = 1, \dots, n$. We assume $\lambda_1 \geq \dots \geq \lambda_r \geq \lambda_{r+1} = 0 = \dots = \lambda_n$

$$\forall u \quad x^T M x \geq 0$$

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3) let $\sigma_i = \sqrt{\lambda_i}$ and $u_i = \dfrac{Av_i}{\sigma_i}$ for all $i = 1, \dots, r$

(u_i) o.n.b. of Im A : $\begin{cases} \|u_i\|^2 = \dfrac{1}{\sigma_i^2} v_i^T A^T A v_i = \dfrac{\lambda_i}{\sigma_i^2} \|v_i\|^2 = 1 \\ \langle u_i, u_j \rangle = \dfrac{\lambda_j}{\sigma_i \sigma_j} v_i^T v_j = 0 \quad \text{if } i \neq j \end{cases}$

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4) Hence $U_1 := (u_1, \dots, u_r) = AV_1\Sigma_1^{-1}$, where $\Sigma_1 = \mathrm{diag}(\sigma_1, \dots, \sigma_r)$

$V_1 = (v_1, \dots, v_r)$

i.e. $U_1\Sigma_1 = AV_1$

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↳ $U_1\begin{pmatrix}\Sigma_1 & 0\end{pmatrix} = AV$ (since $v_{r+1}, \dots, v_n \in \ker A^TA = \ker A$)

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↳ $U \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} = AV$ where $(u_1, \dots, u_{\underline{m}})$ o.n.b. of $\mathbb{R}^m$

SVD, RANGE, KERNEL

"left singular vectors"

"right singular vectors"!

$$A = \Big(\underbrace{u_1 \;\vdots\; \cdots \;\vdots\; u_r}_{\text{o.n.b } \operatorname{Im} A} \;\vdots\; \underbrace{u_{r+1} \;\cdots\; \vdots\; u_m}_{\substack{\text{o.n.b } \operatorname{Ker} A^T \\ = (\operatorname{Im} A)^{\perp}}} \Big) \begin{array}{cc} \left(\begin{array}{cc} \begin{matrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{matrix} & 0 \\ 0 & 0 \end{array}\right) & \begin{matrix} \updownarrow r \\ \updownarrow m-r \end{matrix} \\ \begin{matrix} \longleftrightarrow \\ r \end{matrix} \quad \begin{matrix} \longleftrightarrow \\ m-r \end{matrix} & \end{array} \begin{pmatrix} v_1^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_m^T \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{o.n.b } \operatorname{Im} A^T = (\operatorname{Ker} A)^{\perp} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{o.n.b } \operatorname{Ker}(A^T A) = \operatorname{Ker} A \end{matrix}$$

$$E = \operatorname{Ran} A + (\operatorname{Ran} A)^{\perp}$$

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Prop :

- $\operatorname{Ker} A = \operatorname{Span}(v_{r+1}, \dots, v_m)$

  $\operatorname{Ran} A = \operatorname{Span}(u_1, \dots, u_r)$ (in particular rank $A = r$)

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- Singular values are unique (σ_i^2 eigenvalue of A^TA), singular vectors with multiplicity 1 are unique up to ±1 factor

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- $$A = U \Sigma V^T = \underbrace{U_1 \Sigma_1 V_1^T}_{\text{reduced SVD}} = \sum_{i=1}^{\textcircled{r} = \operatorname{rank} A} \sigma_i u_i v_i^T$$

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  $\boxed{r}$ = rank A

  generalizable to Hilbert bases of $\overline{\operatorname{Ran} A}$ and $(\operatorname{Ker} A)^\perp$

CONDITION NUMBER

Prop : let $A = U \begin{pmatrix} \sigma_1 \ddots & & 0 \\ & \sigma_n & \\ 0 & & 0 \end{pmatrix} V^T$, $\sigma_1 \geqslant \dots \geqslant \sigma_n > 0$. Then $\|A\|_{2,2} = \sigma_1$

Proof : let $x \in \{x \in \mathbb{R}^m / \|x\|_2 = 1\}$. Then

$$\|Ax\|_2^2 = \|U \Sigma V^T x\|_2^2 = \|\Sigma V^T x\|_2^2 \quad \text{since } U \text{ orthogonal}$$

$$= \sum_{i=1}^{n} \sigma_i^2 |v_i^T x|^2$$

$$\leqslant \sigma_1^2 \sum_{i=1}^{n} |v_i^T x|^2 \quad (= \text{ if } x = v_1)$$

$$= \sigma_1^2 \|V^T x\|^2 = \sigma_1^2 \|x\|^2 = \sigma_1^2$$

$$\sup_{\|x\|_2 = 1} \|Ax\|_2 = \|A\|_{2,2}$$

Hence $\boxed{\|A\|_{2,2} = \sup_{\|x\|_2=1} \|Ax\|_2 = \sigma_1}$

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Hence $\boxed{\|A\|_{2,2} = \sup_{\|x\|_2 = 1} \|Ax\|_2 = \sigma_1}$

Prop : let $A \in \mathbb{R}^{m \times m}$ invertible. Then $K_2(A) = \dfrac{\sigma_1}{\sigma_m}$

↖ $A^{-1} = V \Sigma^{-1} U^T$

$$A = U \Sigma V^T \longrightarrow A^{-1} = V \Sigma^{-1} U^T$$

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Rmk : If $A \in \mathbb{R}^{m \times m}$, $\mathrm{rank}(A) = n$, $\dfrac{\sigma_1}{\sigma_n} = K\left(P_{\mathrm{Ran} A} \, A\big|_{(\mathrm{Ker} A)^{\perp}} \right)$

INTERPRETATION OF $K(A)$

relative error on solution $\simeq K(A) \cdot$ relative error on data

e.g. ↳ $10^{-a} \simeq K(A) \cdot 10^{-b}$

↳ $a \simeq b - \log_{10}(K(A))$

$\log_{10}(K(A))$ = number of significant digits one may lose when solving $Ax = y$.

SOLVING (LS) WITH SVD

$$\min_{x \in \mathbb{R}^m} \|Ax - y\|_2^2 \qquad (LS)$$

Let (U, Σ, V) be the SVD of A. We have, for $x \in \mathbb{R}^m$:

$$\|Ax - y\|_2^2 = \|U\Sigma V^T x - y\|_2^2$$

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$$= \sum_{i=1}^{r} |\sigma_i (v_i^T x) - (u_i^T y)|^2 + \sum_{i=r+1}^{m} |u_i^T y|^2$$

because $\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots \; \sigma_r & \\ 0 & & 0 \end{pmatrix} \rightarrow$

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$$= \underbrace{\sum_{i=1}^{r} \left|\sigma_i (v_i^T x) - (u_i^T y)\right|^2}_{= 0 \text{ when } v_i^T x = \frac{u_i^T y}{\sigma_i}} + \underbrace{\sum_{i=r+1}^{m} \left|u_i^T y\right|^2}_{\text{does not depend on } x}$$

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$\Rightarrow$ The minimum is attained for

$$x_* = \underbrace{\sum_{i=1}^{r} \frac{u_i^T y}{\sigma_i} v_i}_{= x_+ \in (\operatorname{Ker} A)^\perp} + \underbrace{x_0}_{\in \operatorname{Ker} A}$$

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$$\min_{x\in\mathbb{R}^m} \|Ax-y\|_2^2 \qquad (LS)$$

let (U,Σ,V) be the SVD of A. We have, for $x\in\mathbb{R}^m$:

$$\|Ax-y\|_2^2 = \|U\Sigma V^T x - y\|_2^2 = \|\Sigma V^T x - U^T y\|_2^2$$

because $\Sigma = \begin{pmatrix}\sigma_1 & & 0\\ & \ddots \; \sigma_r & \\ 0 & & 0\end{pmatrix} \rightarrow$

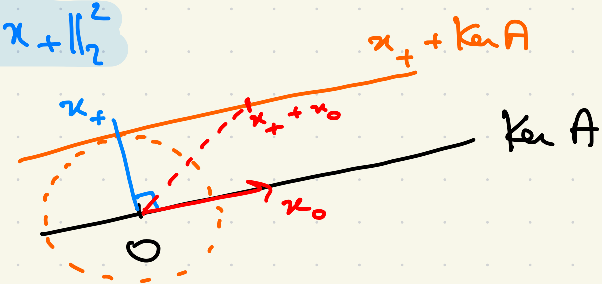
$$= \underbrace{\sum_{i=1}^{r} \left|\sigma_i(v_i^T x) - (u_i^T y)\right|^2}_{=0 \text{ when } v_i^T x = \frac{u_i^T y}{\sigma_i}} + \underbrace{\sum_{i=r+1}^{m} |u_i^T y|^2}_{\text{does not depend on } x}$$

$\Rightarrow$ The minimum is attained for

$$x_* = \underbrace{\sum_{i=1}^{r} \frac{u_i^T y}{\sigma_i} v_i}_{= x_+ \in (\mathrm{Ker}\,A)^\perp} + \underbrace{x_0}_{\in \mathrm{Ker}\,A}$$

and $\|x_*\|_2^2 = \|x_+\|_2^2 + \|x_0\|_2^2$

$\geq \|x_+\|_2^2$



PSEUDO-INVERSE

- Solutions of $\min_{x \in \mathbb{R}^n} \|Ax - y\|_2^2$ are of the form

$$x_* = x_+ + x_0, \quad \text{where} \quad x_+ = \sum \frac{u_i^T y}{\sigma_i} v_i \quad \text{and} \quad x_0 \in \ker A$$

- In matrix form :

$$\mathbb{R}^n \ni x_+ = A^\dagger y, \quad \text{where} \quad A^\dagger := V \begin{array}{cc} & \begin{array}{cc} r & m-r \end{array} \\ \begin{array}{c} r \\ n-r \end{array} & \begin{pmatrix} \begin{matrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_r \end{matrix} & 0 \\ 0 & 0 \end{pmatrix} \end{array} U^T \in \mathbb{R}^{n \times m}$$

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Prop :
- If A is injective, then A^+y is the (unique) solution of the least-squares problem.
- If A is not injective, then A^+y is the (unique) solution of minimal ℓ^2-norm.

Rem : $\frac{\sigma_1}{\sigma_r} = \|A\| \, \|A^+\| =: \kappa_2(A)$ for non-invertible matrix

PSEUDO-INVERSE (2)

- More generally

Def: let $A \in \mathcal{L}(E,F)$, and let $\tilde{A} : (\operatorname{Ker} A)^{\perp} \longrightarrow \operatorname{Ran} A$ be its restriction. The Moore-Penrose pseudo-inverse A^{+} (or generalized inverse) is the unique linear extension of $\tilde{A}^{-1}$ to $\underbrace{\operatorname{Ran} A \oplus (\operatorname{Ran} A)^{\perp}}_{= F}$ with $\operatorname{Ker} A^{+} = (\operatorname{Ran} A)^{\perp}$.

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- **Prop** : A^{+} is characterized by the equations
  - $AA^{+}A = A$
  - $A^{+}AA^{+} = A^{+}$
  - $A^{+}A = P_{(\operatorname{Ker} A)^{\perp}} \left(= I - P_{\operatorname{Ker} A}\right)$
  - $AA^{+} = P_{\operatorname{Ran} A}$

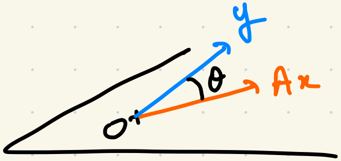
CONDITIONING OF LEAST-SQUARES

for $m \geq n$

Thm: let $A \in \mathbb{R}^{m \times n}$ with rank $A = n$. let x be the solution of $\min_z \|Az - y\|_2^2$, and let $\tilde{x}$ be the solut° of the perturbed problem $\min_z \|Az - (y + \delta y)\|_2^2$. Then

$$\frac{\|x - \tilde{x}\|_2}{\|x\|_2} \lesssim \frac{\kappa(A)}{\cos\theta} \frac{\|\delta y\|_2}{\|y\|_2}$$

where $\theta := \text{angle}(y, Ax)$



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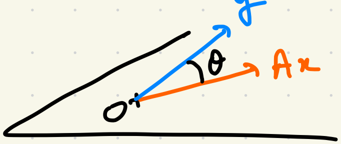
let $\hat{x}$ be the solution of $\min_z \|(A + \delta A)z - (y + \delta y)\|_2^2$

Then

$$\frac{\|x - \hat{x}\|_2}{\|x\|_2} \lesssim \kappa(A)\left(\frac{\|\delta A\|_2}{\|A\|_2} + \frac{1}{\cos\theta}\frac{\|\delta y\|_2}{\|y\|_2}\right) + \kappa(A)^2 \, \mathrm{tg}\,\theta \, \frac{\|\delta A\|_2}{\|A\|_2}$$



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<u>Proof</u> (of the 1st inequality) : let $\delta x = \tilde{x} - x$

normal equations : $A^T A \tilde{x} = A^T(y + \delta y)$

$$A^T A x = A^T y$$

hence $A^T A \, \delta x = A^T \delta y$

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Since A is full column-rank$^{(*)}$, A^TA is invertible and

$$\delta x = (A^TA)^{-1}A^T\delta y$$

↳ $\|\delta x\|_2 \leq \|(A^TA)^{-1}A^T\|_2 \, \|\delta y\|_2$

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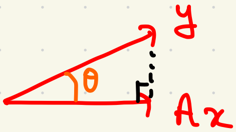

$$\leq \kappa(A)\frac{\|\delta y\|_2}{\|y\|_2}\frac{\|y\|_2}{\|Ax\|_2} = \boxed{\frac{\kappa(A)}{\cos\theta}\frac{\|\delta y\|_2}{\|y\|_2}}$$

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  If $A^TA > 0$ then $\exists!\, R$ upper triang. with $R_{ii} > 0$ s.t. $A^TA = R^TR$

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Example :

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, \quad A^TA = \begin{bmatrix} 1+\varepsilon^2 & 1 & 1 \\ 1 & 1+\varepsilon^2 & 1 \\ 1 & 1 & 1+\varepsilon^2 \end{bmatrix}$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ \varepsilon x_1 \\ \varepsilon x_2 \\ \varepsilon x_3 \end{bmatrix}$$

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  reasonably stable

A GLANCE IN INFINITE DIMENSION

let $A \in \mathcal{L}(E,F)$ continuous

- Properties in finite dimension
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let $(e_1, \dots, e_d)$ basis of V. Then $(e_1, \dots, e_d, x)$ lin. indep. in E, so in $\text{Span}(e_1, \dots, e_d, x)$,

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Why does it matter ?

1) maintain Hilbert structure (completeness)

2) existence of projection

$$\|z - x_0\| = \inf_{x \in S} \|z - x\|$$

("projection theorem")

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$$\overline{\operatorname{Ran} A} \oplus \underset{\overline{\operatorname{Ran} A}^{\perp}}{\overset{}{\underset{\|}{\operatorname{Ran} A^{\perp}}}}$$

(exercise)

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Rmk: $\operatorname{Ran} A \oplus (\operatorname{Ran} A)^{\perp}$ is dense in F

COMPACT OPERATORS

- SVD can be generalized for a specific class of operators

Def: let $A \in \mathcal{L}(E, F)$. A is said to be compact if for any bounded set $B \subset E$, $\overline{A(B)}$ is compact.

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- Ubiquitous in linear models, lead to ill-posedness

Examples :
- $Af = \int_\Omega k(\cdot - y) f(y)\, dy$, with e.g. $k(x) = e^{-\frac{|x|^2}{\sigma^2}}$ (convolution)
- More generally, $Af = \int_\Omega K(\cdot, y) f(y)\, dy$

COMPACT OPERATORS

- SVD can be generalized for a specific class of operators

Def : let $A \in \mathcal{L}(E,F)$. A is said to be compact if for any bounded set $B \subset E$, $\overline{A(B)}$ is compact.

- Ubiquitous in linear models, lead to ill-posedness

Examples :
- $Af = \int_\Omega k(\cdot - y)\, f(y)\, dy$, with e.g. $k(x) = e^{-\frac{|x|^2}{\sigma^2}}$ (convolution)
- More generally, $Af = \int_\Omega K(\cdot, y)\, f(y)\, dy$

* ill-posedness $\longleftrightarrow$ "smoothness" of the kernel K

* in fact

Prop : If E, F are of infinite dimension, then $A \in \mathcal{L}(E,F)$ compact is never invertible.

SINGULAR VALUE EXPANSION

- SVD can be generalized to compact operators

Thm (Singular Value Expansion) Let $A \in \mathcal{L}(E,F)$ be compact.

Then $\exists \, (\sigma_j)_{j \in \mathbb{N}} \in \mathbb{R}_+$, and orthon. fam. $(e_j) \in E$ and $(f_j) \in F$ s.t. $\sigma_j \underset{j\to\infty}{\searrow} 0$ and

$$\begin{cases} \forall x \in E, \quad Ax = \sum_{j=1}^{\infty} \sigma_j \langle x, e_j \rangle f_j \\ \forall y \in F, \quad A^* y = \sum_{j=1}^{\infty} \sigma_j \langle y, f_j \rangle e_j \end{cases}$$

(f_j) Hilbert basis* of $\overline{\operatorname{Ran} A}$

(e_j) Hilbert basis of $(\operatorname{Ker} A)^{\perp}$

* 1) orthonormal

2) Span (e_i) dense

$$\left(x \mapsto \sqrt{2}\cos(2nx)\right) \cup \left(x \mapsto \sqrt{2}\sin(2nx)\right) \qquad n \in \mathbb{N}$$

$$f \in L^2([0,\pi])$$

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Importantly, $\| \sum_1^N \sigma_j \langle x, e_j \rangle f_j \|^2 \leq \sum_1^N \sigma_j^2 \langle x, e_j \rangle^2 \leq \sigma_1^2 \sum_1^N \langle x, e_j \rangle^2 \underset{\text{Bessel}}{\leq} \sigma_1^2 \|x\|^2$

so $\lim_{N\to\infty}$ is valid.

PICARD CRITERION

Thm : $A \in \mathcal{L}(E,F)$ compact, with SVE $\{\sigma_i, e_i, f_i\}$

The equation $Ax = y$ has a solution iff

1) $y \in \overline{\operatorname{Ran} A}$

2) $\displaystyle\sum_{j=1}^{\infty} \frac{\langle y, f_j\rangle^2}{\sigma_j^2} < \infty$ (Picard's criterion)

Proof : • If $Ax = y$ then $y \in \operatorname{Ran} A \subset \overline{\operatorname{Ran} A}$. Furthermore

$$\langle x, e_i\rangle = \frac{1}{\sigma_i}\langle x, A^* f_i\rangle = \frac{1}{\sigma_i}\langle y_i, f_i\rangle$$

so $$\sum_{i=1}^{\infty} \frac{1}{\sigma_i^2}\langle y_i f_i\rangle^2 \leq \|x\|^2 < \infty$$

• If $y \in \overline{\operatorname{Ran} A}$ then $y = \displaystyle\sum_{i=1}^{\infty} \langle y, f_j\rangle f_j$, and

$x = \displaystyle\sum \frac{1}{\sigma_j}\langle y, f_j\rangle e_j$ satisfies $Ax = y$.

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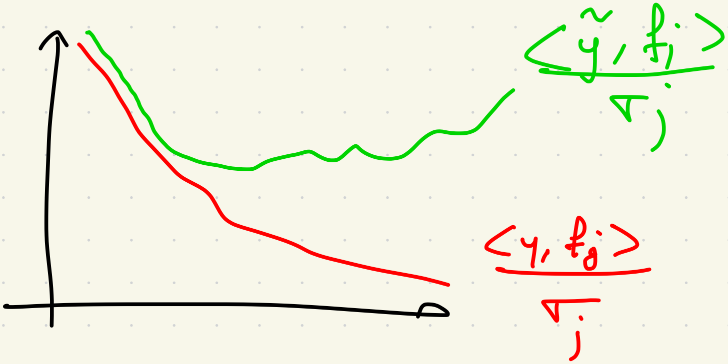
Picard's criterion asks for some regularity on the data : the coefficients $\langle y, f_j \rangle$ must decrease faster than σ_j

Never holds in practice, since $\tilde{y} = y + \delta y$



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