Mathematical Methods for Inverse Problems

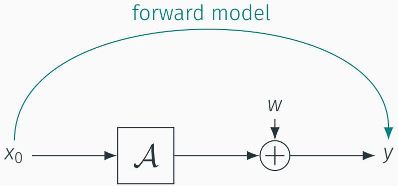
Fallstudien der Mathematischen Modellbildung, Teil 2

20.10.2023 - 21.11.2023,
paul.catala@tum.de

Organisation of the course

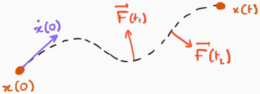
- 10 Vorlesungen, 5 Übungen
- Vorlesungen: Montags 16.15 – 17.45 Uhr und Mittwochs 17 – 18.30 Uhr
- Übungen: Dienstags 12.15 – 13.45 Uhr
- Bewertung: Hausarbeit in Gruppen von Drei. Gruppen müssen per e-mail sich bis zum 31.1.24 anmelden. Abgabetermin: 15.03.24.
- für Fragen: paul.catala@tum.de , Büro 02.08.038

- Direct problem: determine the observations (*i.e.* results of measurements) y given the input x_0 and the parameters of a model $\mathcal{A}$ (and possible noise w).



- Mathematical models based on physical laws allow to predict such measurements.
- Similar inputs x_0 produce similar measurements y.

- Mechanics: given x_0, v_0, $\sum F_i$ and m, determine the position $x(t)$ obeying



$$m\ddot{x} = \sum F_i, \quad \dot{x}(0) = v_0, \quad x(0) = x_0$$

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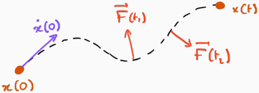

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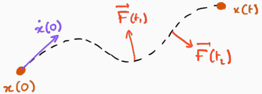
- Optics: given a source distribution and a diffracting object or aperture, determine the light intensity on a screen

Examples of Direct Problems

- Mechanics: given x_0, v_0, $\sum F_i$ and m, determine the position $x(t)$ obeying



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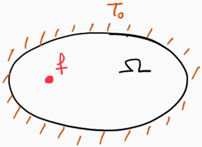
- Optics: given a source distribution and a diffracting object or aperture, determine the light intensity on a screen

- Thermodynamics: Given a domain Ω with thermic conductivity κ, whose boundary is maintained at temperature T_0, and a heat source f, determine the temperature T in Ω in the stationary regime

$$\begin{cases} -\operatorname{div}(\kappa \nabla T) = f & \text{in} \quad \Omega \\ T = T_0 & \text{on} \quad \partial\Omega \end{cases}$$



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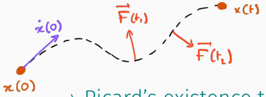

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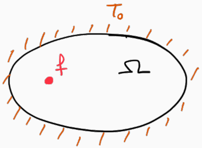

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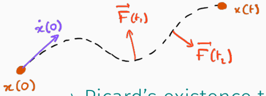

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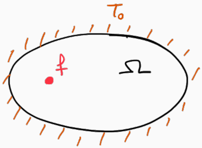

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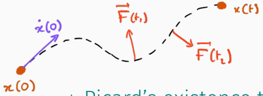

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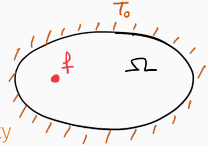


→ Lax-Milgram theorem for elliptic EDP ensures existence and unicity

Well-Posed Problems

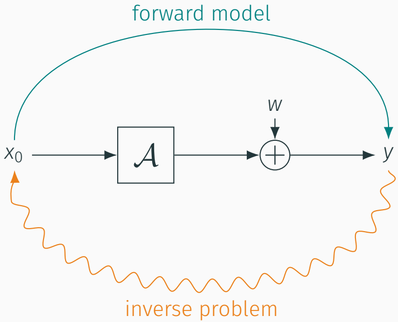
- Direct problems are usually well-posed!

Definition (Hadamard 1923). A problem is well-posed[1] when:

- there is a solution (existence)
- there is at most one solution (unicity)
- the solution depends continuously on the data (stability)

[1] *Hadamard, 1923,* Lectures on the Cauchy Problem in Linear Partial Differential Equations

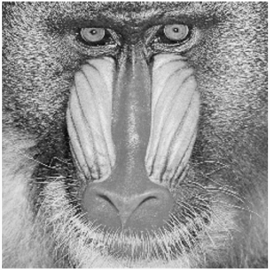
- Inverse problem: determine the source x_0 given indirect, incomplete and possibly noisy observations $y = \mathcal{A}(x_0) + w$.



- Inverse problems are usually ill-posed
  - experimental data are noisy, there can be model mismatches: ~~existence~~
  - different parameters may lead to the similar measurements: ~~unicity~~, ~~stability~~

Deconvolution

convolution



$x_0(u)$

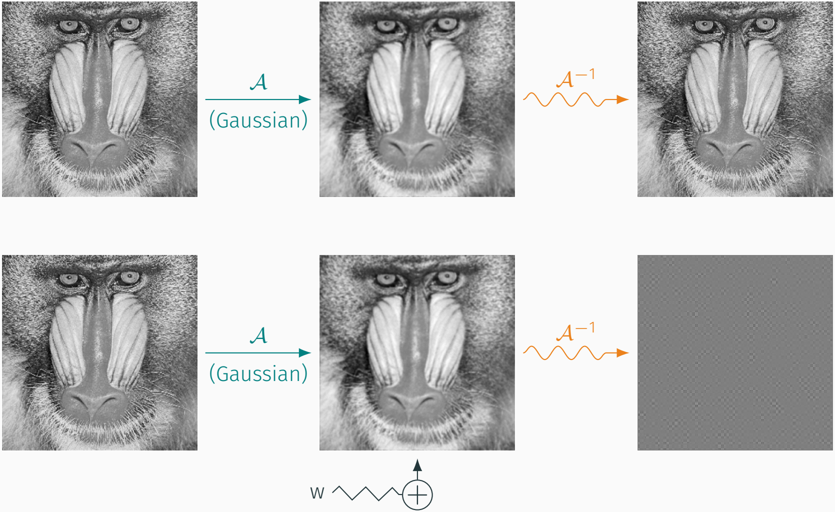
$y_0(u) = \sum_{u'} a(u - u')x_0(u')$

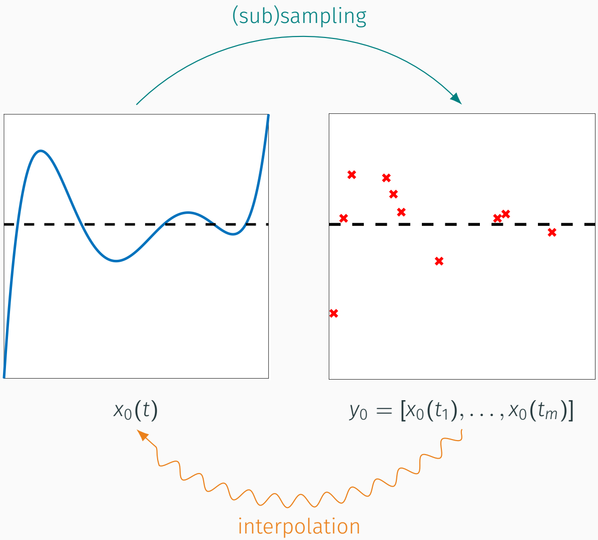
deconvolution

Inverse Problems are Usually Ill-Posed

Example (Deconvolution). If $y = a * x =: \mathcal{A}(x)$, then in Fourier $\hat{y} = \hat{a} \cdot \hat{x}$. Assuming that the support of $\hat{a}$ is sufficiently large, a solution of the deconvolution problem is given by

$$x = \mathcal{F}^{-1}(\hat{y} \oslash \hat{a}) =: \mathcal{A}^{-1}(y).$$

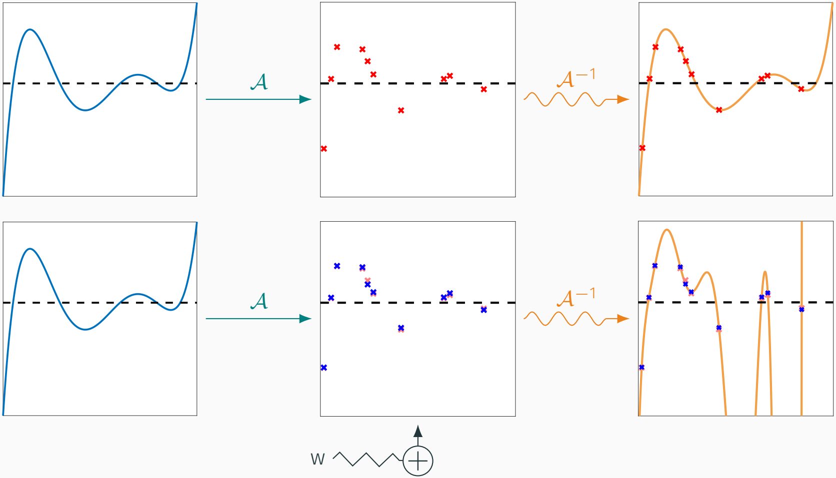


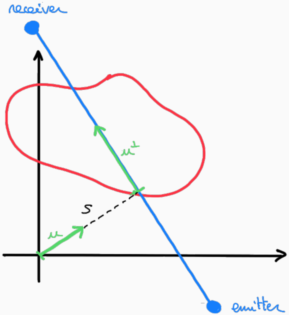

(sub)sampling
$x_0(t)$
$y_0 = [x_0(t_1), \ldots, x_0(t_m)]$
interpolation

Example (Interpolation). If samples $\{(x_j, y_j)\}_{0 \leqslant j \leqslant n}$ are given, then

$$\tilde{p}(x) := \sum_{j=1}^{n} y_j \prod_{i=0, i \neq j}^{n} \frac{x - x_i}{x_j - x_j}$$

is the only polynomial of degree at most n that takes the value y_j at x_j.





- X-rays travel in straight lines, parameterized via normal vector $u \in \mathbb{R}^2$ and distance to the origin s
- attenuation proportional to intensity $\mathcal{I}$ itself, and distance covered Δt

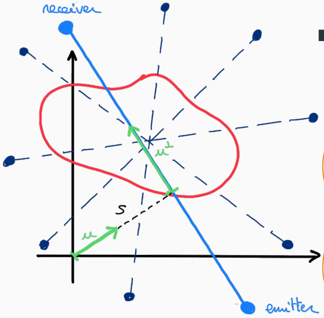
$$\Delta\mathcal{I}(su + tu^\perp) = -\mathcal{I}(su + tu^\perp)f(su + tu^\perp)\Delta t$$

$\Delta t \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(su + tu^\perp) = -\mathcal{I}(su + tu^\perp)f(su + tu^\perp)$$

integrate between emitter ($-\infty$) and receiver ($+\infty$)

$$-\ln\frac{\mathcal{I}_L(s,u)}{\mathcal{I}_0(s,u)} = \int_{\mathbb{R}} f(su + tu^\perp)\mathrm{d}t =: \mathcal{R}f(s,u)$$



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f —Radon transform→ $\mathcal{R}f(s,u)$

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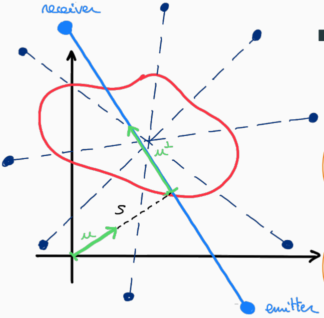

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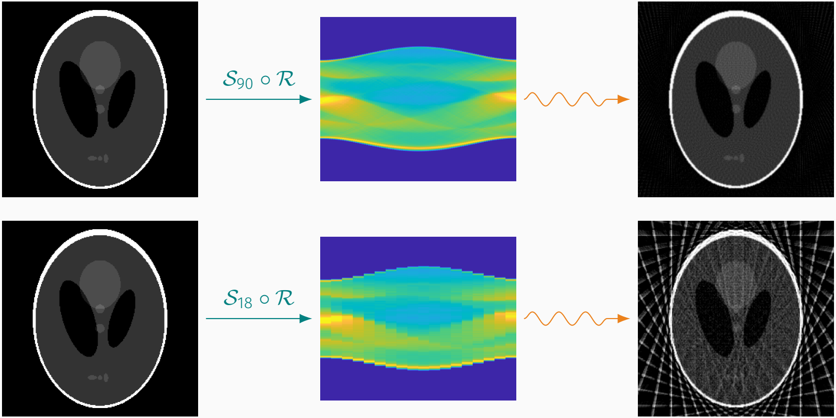
$\mathcal{R}f(s_1, u_1), \ldots, \mathcal{R}f(s_m, u_m)$ ⇝ f: ??

Inverse Problems are Usually Ill-Posed

Example (Radon transform).



"depletion"

	u1	u2	u3	...
Love Actually	1	1	-1	...
The Matrix	-1	-1	-1	...
Notting Hill	1	1	-1	...
⋮	⋮	⋮	⋮	

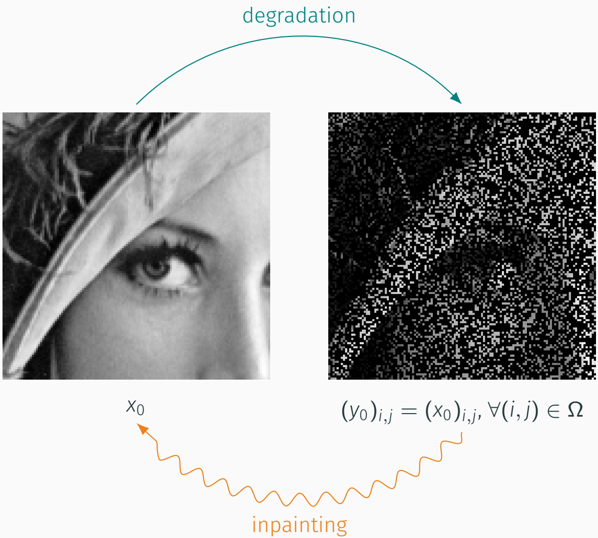
x_0

	u1	u2	u3	...
Love Actually	1	1	?	...
The Matrix	?	-1	?	...
Notting Hill	?	?	-1	...
⋮	⋮	⋮	⋮	

$(y_0)_{i,j} = (x_0)_{i,j}, \forall (i,j) \in I$

completion

Interpolation



- Inverse problems arise in[1,2]
    - microscopy,
    - medical imaging,
    - hydrogeology,
    - chemistry,
    - radar,
    - quantum mechanics, ...

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  → instability can be tackled via *regularization* of the problem

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[2] *Engl, Hanke, and Neubauer, 1996*, Regularization of Inverse Problems

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  Inverse problem $\Leftrightarrow$ solving $\mathcal{A}(x) = y$ given y in a data space F.

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- E, F are typically Hilbert spaces (*e.g.* $E = F = \mathrm{L}^2(\Omega)$, $\Omega \subset \mathbb{R}^2$ in imaging), can even be Banach spaces[1] (*e.g.* space of measures[2]) .

  In this course, $E = \mathbb{R}^n$ (x_0 models coefficients wrt *e.g.* standard basis, Fourier basis, wavelet basis) and $F = \mathbb{R}^m$ (y is the data vector, or feature vector).

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[2] *Bredies and Pikkarainen, 2013,* Inverse problems in spaces of measures

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- $\mathcal{A}$ may be linear or non-linear

  In this course, $\mathcal{A} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ may be seen as a matrix $A \in \mathbb{R}^{m \times n}$, and $y = Ax$.

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Focus of the course

- Given $A \in \mathcal{L}(E, F)$, $y \in F$, solve

$$Ax = y, \quad x \in E$$

- General study via the singular value decomposition
- ℓ^2-regularization: Tikhonov regularization, spectral truncation, algorithms
- sparse regularization: ℓ^0 and ℓ^1-regularization, sparsity wrt dictionary, algorithms