

Exercises 5

Exercise 1 (Maximum likelihood estimation). 1. Let $x \in \mathbb{R}$, and $y_i = x + w_i$ where $w_i \sim \mathcal{N}(0, 1)$. Give the Maximum Likelihood Estimator of x , *i.e.*

$$\hat{x} = \operatorname{argmax}_x p(y|x)$$

2. Same question assuming now a multiplicative Gaussian noise, *i.e.* $y_i \sim xw_i$ with $w_i \sim \mathcal{N}(0, 1)$.

Exercise 2. Let $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^m$, $\eta > 0$ and let $\|\cdot\|$ be an arbitrary norm on \mathbb{R}^m . Show that the solution of the optimization problem

$$x_* = \operatorname{argmin} \|z\|_1 \quad \text{s.t.} \quad \|Az - y\| \leq \eta$$

is m -sparse in the case of the uniqueness of the solution. *Hint:* show that the system of columns $\{a_j ; j \in \operatorname{Supp} x_*\}$ is linearly independent.

Exercise 3 (Null Space Property). 1. Prove the uniform recovery theorem: every k -sparse x_0 is the unique solution of

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax = Ax_0. \tag{BP}$$

if and only if A satisfies the Null Space Property of order k , *i.e.* :

$$\forall I : |I| \leq k, \quad \forall h \in \operatorname{Ker} A \setminus 0, \quad \|h_I\|_1 < \|h_{I^c}\|_1$$

2. Show that if NSP(k) holds, then the solution of (BP) is also a solution of

$$\min \|x\|_0 \quad \text{s.t.} \quad Ax = Ax_0$$

3. Let $x_0 \in \mathbb{R}^n$ (not necessarily k -sparse), let $y = Ax_0 + w$, $\|w\| \leq \varepsilon$, and let x be a solution of

$$\min \|x\|_1 \quad \text{s.t.} \quad \|Ax - y\| \leq \varepsilon \tag{BP- ε }$$

The goal of this question is to prove the uniform robust recovery theorem: if A obeys the robust NSP of order k , *i.e.*

$$\exists 0 < \rho < 1, \quad \exists \tau > 0, \quad \forall I : |I| = k, \quad \forall h \in \operatorname{Ker} A \setminus 0, \quad \|h_I\|_1 \leq \rho \|h_{I^c}\|_1 + \tau \|Ah\|_2$$

then for all $x_0 \in \mathbb{R}^n$, any solution of (BP- ε) satisfies

$$\|x - x_0\|_1 \leq 2 \frac{1 + \rho}{1 - \rho} \sigma_k(x_0)_1 + 4 \frac{\tau}{1 - \rho} \varepsilon$$

where $\sigma_k(x_0)_1 := \inf \{\|x_0 - z\|_1 ; \|z\|_0 \leq k\}$ is the best k -sparse approximation with respect to the ℓ^1 -norm. We assume that A satisfies the robust NSP of order k .

(a) Let $h = x - x_0$. Show that, for any subset I ,

$$\|x_0\|_1 + \|h_{I^c}\| \leq 2\|(x_0)_{I^c}\|_1 + \|h_I\|_1 + \|x\|_1$$

(b) Deduce that for a well chosen subset I ,

$$\|h_{I^c}\|_1 \leq \frac{1}{1-\rho}(2\sigma_k(x_0)_1 + 2\tau\varepsilon)$$

(c) Conclude.