Exercises 4

Exercise 1. Show that, for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$,

$$\min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad a^\top x = b$$

always has a one-sparse vector as minimizer. (This is perhaps the simplest illustration of the general phenomenon that solutions of undetermined ℓ^1 minimization problems tend to be sparse.)

Exercise 2. Compute the dual of the TV denoising problem

$$\min \frac{1}{2}\|x - y\|^2 + \lambda\|Bx\|_1.$$

Exercise 3. 1. Compute the proximal operator of the following functions:

- $f(x) = \mathbb{1}_{Ax=y}(x)$, where A has full row-rank
- $f(x) = \frac{1}{2}\|Ax - y\|^2$
- $f(x) = -\sum \log(x_i)$ (with $x_i > 0$)

2. Solve

$$\min_x \frac{1}{2}\|Bx - z\|_2^2 \quad \text{s.t.} \quad Ax = b$$

assuming that the solution exists and is unique.

Exercise 4 (Elastic Net). The ”Elastic Net” regularized problem combines both ℓ^1 and ℓ^2 regularization. It si given as

$$\min_x \|y - Ax\|^2 + \lambda_2\|x\|_2^2 + \lambda_1\|x\|_1$$

1. What is a suitable decomposition of the objective to perform proximal splitting on this problem?
2. Describe forward-backward updates that can be used to solve it.