# Exercises 3

*Exercise* 1. 1. Let $A \in \mathbb{R}^{m\times n}$, let $I \in \mathbb{R}^{n\times n}$, $\lambda > 0$ and

$$K = \begin{bmatrix} A \\ \sqrt{\lambda} I \end{bmatrix} \in \mathbb{R}^{(m+n)\times n}.$$

Show that the singular values of $K$ satisfy $\lambda_j \geqslant \sqrt{\delta}$, $j = 1, \ldots, n$.

2. Let $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$ and $x_\lambda \in \mathbb{R}^n$ be minimizer of

$$\|Ax - y\|^2 + \lambda \|x\|^2, \quad \delta > 0$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as

$$f(\lambda) = \|Ax_\lambda - y\|^2.$$

Show that

$$f'(\lambda) = 2\lambda \langle x_\lambda,\, (A^\top A + \lambda I)^{-1} x_\lambda \rangle.$$

*Exercise* 2. Let $A \in \mathbb{R}^{m\times n}$ such that $\operatorname{Ker} A = \{0\}$ ($m > n$, and $A$ full column rank). Let $x$ be the solution of $Ax = y$ for some $y \in \operatorname{Ran} A$, and let $y^\delta \in \mathbb{R}^m$ such that $\|y^\delta - y\| \leqslant \delta$.

We define

$$q(\lambda, \sigma) = \begin{cases} 1 & \text{if} \quad \sigma^2 \geqslant \lambda \\ 0 & \text{if} \quad \sigma^2 < \lambda \end{cases}$$

and the operator $R_\lambda$ such that

$$R_\lambda y = \sum_{i=1}^{n} \frac{q(\lambda, \sigma_i)}{\sigma_i} \langle y,\, u_i \rangle v_i = \sum_{\sigma_i^2 \geqslant \lambda} \frac{1}{\sigma_i} \langle y,\, u_i \rangle v_i,$$

where $\{\sigma_i, u_i, v_i\}$ are the singular values and vectors of $A$. Let $x_\lambda^\delta = R_\lambda y^\delta$.

1. Let $\lambda = \lambda(\delta) = c\delta^\theta$ where $c > 0$ and $0 < \theta < 2$. Show that

$$\|x_\lambda^\delta - x\| \to 0 \quad \text{where} \quad \delta \to 0$$

2. Assume that $x = A^\top z$ for some $z \in \mathbb{R}^m$ (in fact this is true because $\operatorname{Ran} A^\top = \operatorname{Ran} A^\dagger = \operatorname{Span}(v_1, \ldots, v_r)$.) Deduce the $\theta$ for which the convergence of $x_\lambda^\delta$ towards $x$ when $\delta \to 0$ is optimal. What is the corresponding rate?

3. Assume that $x = A^\top A w$ for some $w \in \mathbb{R}^n$ (in fact this is true because $\operatorname{Ran} A^\top A = \operatorname{Ran} A^\top = \operatorname{Span}(v_1, \ldots, v_r)$.) Deduce the $\theta$ for which the convergence of $x_\lambda^\delta$ is optimal. What is the corresponding rate?

*Exercise* 3 (Duality).

1. Compute the dual problem of the least-squares problem with Tikhonov regularization.
2. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. We define its dual norm as

$$\forall z \in \mathbb{R}^n, \quad \|z\|_* = \sup\left\{z^\top x \;;\; \|x\| \leqslant 1\right\}$$

   - Show that the dual norm of the Euclidean norm $\|\cdot\|_2$ is the Euclidean norm
   - Show that the dual norm of $\|\cdot\|_\infty$ is $\|\cdot\|_1$.
   - Show that the dual norm of $\|\cdot\|_1$ is $\|\cdot\|_\infty$.