Exercise 2

Exercise 1. 1. Compute the singular value decomposition of (some of) the following matrices

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2. With $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, use the SVD of A to draw the set $\left\{x \in \mathbb{R}^2 \; ; \; \|Ax\|_2 = 1\right\}$. How can you determine the singular values and right singular vectors of A from this figure?

Exercise 2. Let $A \in \mathbb{R}^{m\times n}$ and $A = U\Sigma V^\top$ its singular value decomposition. We write u_i and v_i the left and right singular vectors respectively, and $\sigma_1 \geqslant \ldots \geqslant \sigma_r > 0$ the non-zero singular values.

1. We have seen that $\|A\|_{2,2} := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1$. Show that

$$\sigma_r = \inf_{x \in (\mathrm{Ker}\, A)^\perp \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_2}$$

2. For $k < r$, we define

$$A_k := \sum_{i=1}^{k} \sigma_i u_i v_i^\top,$$

i.e. $A_k = U\Sigma_k V^\top$, where Σ_k is obtained from Σ by setting $\sigma_{k+1} = \ldots = \sigma_r = 0$.

- Show that $\|A - A_k\|_{2,2} = \sigma_{k+1}$.
- Show that A_k actually minimizes $\|A - B\|_{2,2}$ among all $B \in \mathbb{R}^{m\times n}$ such that $\mathrm{rank}\, B \leqslant k$ (this result is known as the Eckart-Young-Mirsky theorem).

Exercise 3. We define the circular convolution of $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ as the vector $(a * x) \in \mathbb{R}^n$ whose entries are given by

$$\forall k \in \{1, \ldots, n\}, \quad (a * x)_k := \sum_{i=1}^{n} a_{[k-i]} x_i$$

where $[i] = i \pmod{n}$. For $a \in \mathbb{R}^n$, let $A : \mathbb{R}^n \mapsto \mathbb{R}^n, x \mapsto a * x$.

1. Write the matrix of A.

2. Let $F \in \mathbb{C}^{n \times n}$ be the DFT matrix, given by

$$F_{kl} = \exp\left(-\frac{2\imath\pi kl}{n}\right), \quad \forall 0 \leqslant k, l \leqslant n-1$$

. Show that

$$AF = \mathrm{Diag}(\hat{a})F,$$

where $\hat{a} := Fa$ is the discrete Fourier transform of a. Deduce the singular values σ_j of A.

3. Let $a = \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \end{bmatrix}^\top$. Compute the condition number of A in that case.

Exercise 4 (Pseudo-inverse). Let $A \in \mathbb{R}^{m \times n}$ and $A^\dagger$ its Moore-Penrose pseudo-inverse.

1. Check the identities

$$A^\dagger A A^\dagger = A^\dagger,$$
$$A A^\dagger A = A,$$

2. Show that $A^\dagger A$ and $AA^\dagger$ are orthogonal projections on $(\mathrm{Ker}\, A)^\perp$ and $\mathrm{Ran}\, A$ respectively.