Exercise 2

Exercise 1. 1. Compute the singular value decomposition of (some of) the following matrices

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2. With $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, use the SVD of A to draw the set $\{x \in \mathbb{R}^2 \; ; \; \|Ax\|_2 = 1\}$. How can you determine the singular values and right singular vectors of A from this figure?

Solution.

1.
   - $A_1^\top A_1 = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$, hence $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   - $A_2^\top A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, diagonalisable with eigenvalues $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and eigenvectors $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, hence $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
   - $A_3^\top A_3 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$, hence $A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
   - $A_4^\top A_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, diagonalisable with eigenvalues $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and eigenvectors $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$, hence $A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$

2. We have $A^\top A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$ diagonalisable with eigenvalues $\begin{bmatrix} 16 & 0 \\ 0 & 4 \end{bmatrix}$ and (normalized) eigenvectors $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, hence $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = U\Sigma U^\top$.

$$\|Ax\|_2^2 = 1 \Leftrightarrow 4(u_1^\top x)^2 + 2(u_2^\top x)^2 = 1$$

In the basis (u_1, u_2), this is the equation of an ellipse of semi-axis $1/2$ (along u_2) and $1/4$ (along u_1).

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Exercise 2. Let $A \in \mathbb{R}^{m\times n}$ and $A = U\Sigma V^\top$ its singular value decomposition. We write u_i and v_i the left and right singular vectors respectively, and $\sigma_1 \geqslant \ldots \geqslant \sigma_r > 0$ the non-zero singular values.

1. We have seen that $\|A\|_{2,2} := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1$. Show that

$$\sigma_r = \inf_{x\in(\operatorname{Ker} A)^\perp\setminus\{0\}} \frac{\|Ax\|_2}{\|x\|_2}$$

2. For $k < r$, we define

$$A_k := \sum_{i=1}^k \sigma_i u_i v_i^\top,$$

i.e. $A_k = U\Sigma_k V^\top$, where Σ_k is obtained from Σ by setting $\sigma_{k+1} = \ldots = \sigma_r = 0$.

- Show that $\|A - A_k\|_{2,2} = \sigma_{k+1}$.
- Show that A_k actually minimizes $\|A - B\|_{2,2}$ among all $B \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank} B \leqslant k$ (this result is known as the Eckart-Young-Mirsky theorem).

Solution.

1. We have, for any $x \in (\operatorname{Ker} A)^\top$,

$$\begin{aligned}
\|Ax\|_2^2 = \|U\Sigma V^\top x\|_2^2 = \|\Sigma V^\top x\|^2 \\
&= \sum_{i=1}^r \sigma_i^2 \langle v_i,\, x\rangle^2 \\
&\geqslant \sigma_r^2 \sum_{i=1}^r \langle v_i,\, x\rangle^2 \\
&= \sigma_r^2 \|V^\top x\|_2 \quad (\text{since } x \in (\operatorname{Ker} A)^\top,\ x \in \operatorname{Span} 0(v_1, \ldots, v_r)) \\
&= \sigma_r \|x\|_2
\end{aligned}$$

2.
- We have

$$\begin{aligned}
\|A - A_k\|_{2,2} &= \|U(\Sigma - \Sigma_k)V^\top\|_{2,2} \\
&= \|\Sigma - \Sigma_k\|_2 \\
&= \|\operatorname{Diag}(0, \ldots, \sigma_{k+1}, \ldots, \sigma_r, 0, \ldots)\|_{2,2} \\
&= \sigma_{k+1}
\end{aligned}$$

- Let $B \in \mathbb{R}^{m\times n}$ be of rank at most k. Let $x \in \operatorname{Ker} B \cap \operatorname{Span}(v_1, \ldots, v_{k+1})$. Such a x exists: since

$$\dim \operatorname{Ker} B + \dim \operatorname{Span}(v_1, \ldots, v_{k+1}) \geqslant n - k + k + 1 = n + 1 > n,$$

this implies that $\operatorname{Ker} B \cap \operatorname{Span}(v_1, \ldots, v_{k+1}) \neq \{0\}$. Thus

$$\|Ax - Bx\|^2 = \|Ax\|^2 = \sum_{i=1}^r \sigma_i^2 \langle v_i,\, x\rangle^2 = \sum_{i=1}^{k+1} \sigma_i^2 \langle v_i,\, x\rangle^2 \geqslant \sigma_{k+1}^2 \|x\|^2,$$

and therefore $\|A - B\| \geqslant \|A - A_k\|$ for all B of rank at most k.

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Exercise 3. We define the circular convolution of $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ as the vector $(a * x) \in \mathbb{R}^n$ whose entries are given by

$$\forall k \in \{1, \dots, n\}, \quad (a * x)_k := \sum_{i=1}^{n} a_{[k-i]} x_i$$

where $[i] = i \pmod n$. For $a \in \mathbb{R}^n$, let $A : \mathbb{R}^n \mapsto \mathbb{R}^n, x \mapsto a * x$.

1. Write the matrix of A.
2. Let $F \in \mathbb{C}^{n \times n}$ be the DFT matrix, given by

   $$F_{kl} = \exp\left(-\frac{2\imath\pi kl}{n}\right), \quad \forall 0 \leqslant k, l \leqslant n-1$$

   . Show that

   $$AF = F \operatorname{Diag}(\hat{a}),$$

   where $\hat{a} := Fa$ is the discrete Fourier transform of a. Deduce the singular values σ_j of A.
3. Let $a = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \end{bmatrix}^\top$. Compute the condition number of A in that case.

Solution.

1. We have

   $$A = \begin{bmatrix} a_0 & a_n & \dots & a_1 \\ a_1 & a_1 & \dots & a_2 \\ \vdots & \vdots & & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{bmatrix}$$

2. We have

   $$\begin{aligned} (Af_j)_i &= \sum_{k=0}^{n-1} a_{[i-k]} \exp\left(-\frac{2\imath\pi kj}{n}\right) \\ &= \sum_{k=0}^{n-1} a_k \exp\left(-\frac{2\imath\pi(i-k)j}{n}\right) \\ &= \exp\left(-\frac{2\imath\pi ij}{n}\right) \sum_{k=0}^{n-1} a_k \exp\left(\frac{2\imath\pi kj}{n}\right) \\ &= \exp\left(-\frac{2\imath\pi ij}{n}\right) \hat{a}_j \end{aligned}$$

   hence $Af_j = \hat{a}_j f_j$, and $AF = F \operatorname{Diag}(\hat{a})$.
3. The matrix $\frac{1}{\sqrt{n}} F$ is orthogonal (exercise), and we have

   $$A = \frac{1}{\sqrt{n}} F \operatorname{Diag}(nFa) \frac{1}{\sqrt{n}} F^*.$$

   This does not give us the SVD of A since the coefficients nFa are not real, but the singular values are immediately deduced from the eigenvalues of $A^\top A$, *i.e.* $\sigma_j = n|\hat{a}_j|$.

4. With this particular example we have

$$\hat{a}_k = 1 - \exp\left(-2\imath\pi\frac{k}{n}\right).$$

Hence $|\hat{a}_k|$ is smallest and non-zero for $k = 1/n$ (angle 0), and largest when $k/n \simeq 1/2$ (angle pi), more precisely when $k = \lfloor n/2 \rfloor$. Therefore we have

$$\kappa(A) = \frac{|1 - \exp\left(-\frac{2\imath\pi\lfloor n/2\rfloor}{n}\right)|}{|1 - \exp\left(-\frac{2\imath\pi}{n}\right)|}.$$

Note in particular that $\kappa(A) \to \infty$ if $n \to \infty$.

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Exercise 4 (Pseudo-inverse). Let $A \in \mathbb{R}^{m\times n}$ and $A^\dagger$ its Moore-Penrose pseudo-inverse.

1. Check the identities

$$A^\dagger A A^\dagger = A^\dagger,$$
$$A A^\dagger A = A,$$

2. Show that $A^\dagger A$ and $AA^\dagger$ are orthogonal projections on $(\operatorname{Ker} A)^\perp$ and $\operatorname{Ran} A$ respectively.

Solution.

1. These relations are easily obtained using the singular value decomposition of A.
2.

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