Exercise 1

Exercise 1 (Linear algebra reminders). Let $(E, \langle \cdot, \cdot \rangle_E)$ and $(F, \langle \cdot, \cdot \rangle_F)$ be *finite-dimensional* Hilbert spaces, and let $A \in \mathcal{L}(E, F)$.

1. Let G be a subspace of E. Show that $E = G \oplus G^\perp$.

   Hint: one may introduce a basis of G and use the projection operator on G.

2. Show that

   - $\operatorname{Ker} A^* = (\operatorname{Ran} A)^\perp$
   - $\operatorname{Ran} A^* = (\operatorname{Ker} A)^\perp$
   - $\operatorname{Ker} A^* A = \operatorname{Ker} A$

3. Show that if A has full column rank (hence $\dim E \leqslant \dim F$), then $A^* A$ is invertible.

Exercise 2 (Regression). Given $\tau = \{t_1, \dots, t_m\} \subset \mathbb{R}$, we define

$$\begin{aligned} \mathcal{A}_n^\tau : \mathbb{R}_{n-1}[X] &\to \mathbb{R}^m \\ p &\mapsto \big[p(t_1), \dots, p(t_m)\big]^\top, \end{aligned}$$

and we consider the inverse problem

$$\mathcal{A}_n^\tau(p) = y \tag{1}$$

given some $y \in \mathbb{R}^m$.

1. Show that $\mathcal{A}_n^\tau$ is linear and give its matrix representation A_n^τ with respect to the canonical bases of $\mathbb{R}_{n-1}[X]$ and $\mathbb{R}^m$.

2. $\star$ Suppose $n = m$. Show that $\det(A_m^\tau) = \prod_{i<j}(t_j - t_i)$. When does (1) admit a unique solution in that case?

3. Suppose $n < m$. Why is the problem ill-posed in that case? We consider the least-square formulation

   $$\min_{p \in \mathbb{R}^n} L(p) := \|A_n^\tau p - y\|_2^2. \tag{2}$$

   Show that L is convex, and deduce the normal equations.

4. In this question, we assume that $n = 2$ and $m > n$. Show that the solution of (2) is a line that passes through the arithmetic mean of the points $((t_1, y_1), \dots, (t_m, y_m))$.

   Hint: With $p = (\alpha, \beta) \in \mathbb{R}^2$, consider the partial derivative of $L(\alpha, \beta)$ with respect to α.

Exercise 3. Let $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ with $y_1 \leqslant y_2 \leqslant y_3$. We consider the linear system $Ax = y$ for $x \in \mathbb{R}$.

1. Is this system well-posed? why?

2. Let $p \in [1, +\infty]$. We replace the system by the following problem

$$\min_{x \in \mathbb{R}} \|Ax - y\|_p^p \tag{3}$$

   Compute the solution of (3) for $p = 1, 2, \infty$.

Exercise 4 (An example in infinite dimension). Let $E = \mathrm{L}^2([0,1])$, endowed with the L^2-norm, and let $\mathcal{A}$ be the operator defined by

$$\mathcal{A}f(x) = \int_0^x f(t)\mathrm{d}t$$

1. Check that $A \in \mathcal{L}(E, E)$, and that it is continuous.

2. Show that A is injective.

3. Let $F := \{g \in C^1([0,1])\ ;\ g(0) = 0\}$. Show that $F \subset \operatorname{Ran} A$. This allows to consider the restriction $\mathcal{A}^{-1}|_F : F \to E$ of $\mathcal{A}^{-1} : \operatorname{Ran} A \to E$.

4. Show that $\mathcal{A}^{-1}|_F$ is not continuous.

   Hint: consider the function $f_n(x) = f(x) + \frac{1}{n}\sin(n^2 x)$ for $f \in C^1([0,1])$ with $f(0) = 0$.