Exercise 1

Exercise 1 (Linear algebra reminders). Let $(E, \langle \cdot, \cdot \rangle_E)$ and $(F, \langle \cdot, \cdot \rangle_F)$ be *finite-dimensional* Hilbert spaces, and let $A \in \mathcal{L}(E, F)$.

1. Let G be a subspace of E. Show that $E = G \oplus G^\perp$.

   Hint: one may introduce a basis of G and use the projection operator on G.

2. Show that

   - $\mathrm{Ker}\, A^* = (\mathrm{Ran}\, A)^\perp$
   - $\mathrm{Ran}\, A^* = (\mathrm{Ker}\, A)^\perp$
   - $\mathrm{Ker}\, A^*A = \mathrm{Ker}\, A$

3. Show that if A has full column rank (hence $\dim E \leqslant \dim F$), then A^*A is invertible.

Solution.

1. Let $x \in G \cap G^\perp$. Then $0 = \langle x, x \rangle_E = \|x\|_E^2$, hence $x = 0$. Therefore, $\underline{G \cap G^\perp = \{0\}}$.

   Let $x \in E$. Let $(e_1, \dots, e_k)$ be an orthonormal basis of G, and consider the projection on G:

   $$\forall x \in E, \quad P_G(x) = \sum_{i=1}^{k} \langle x, e_i \rangle e_i$$

   Then $x = P_G(x) + (x - P_G(x))$, and one checks easily that $(x - P_G(x))$ is in $G^\perp$. Hence $x \in G + G^\perp$. This shows $\underline{E = G + G^\perp}$.

2. We only prove the second equality. The inclusion $\mathrm{Ran}\, A^* \subset (\mathrm{Ker}\, A)^\perp$ is straightforward. Let $\dim E = n$, $\dim F = m$. Then, since

   $$\dim \mathrm{Ran}\, A^* = m - \dim \mathrm{Ker}\, A^* = m - \dim(\mathrm{Ran}\, A)^\perp = \dim \mathrm{Ran}\, A = n - \dim \mathrm{Ker}\, A = \dim(\mathrm{Ker}\, A)^\perp$$

   which proves the equality.

3. A full column rank means that $\mathrm{Ker}\, A = \{0\}$, and hence $\mathrm{Ker}\, A^*A = \{0\}$. Hence $\dim \mathrm{Ran}\, A^*A = n$ and $\mathrm{Ran}\, A^*A = F$.

■

Exercise 2 (Regression). Given $\tau = \{t_1, \dots, t_m\} \subset \mathbb{R}$, we define

$$\begin{aligned} \mathcal{A}_n^\tau : \mathbb{R}_{n-1}[X] &\to \mathbb{R}^m \\ p &\mapsto \big[p(t_1), \dots, p(t_m)\big]^\top, \end{aligned}$$

and we consider the inverse problem

$$\mathcal{A}_n^\tau(p) = y \tag{1}$$

given some $y \in \mathbb{R}^m$.

1. Show that $\mathcal{A}_n^\tau$ is linear and give its matrix representation A_n^τ with respect to the canonical bases of $\mathbb{R}_{n-1}[X]$ and $\mathbb{R}^m$.

2. $\star$ Suppose $n = m$. Show that $\det(A_m^\tau) = \prod_{i<j}(t_j - t_i)$. When does (1) admit a unique solution in that case?

3. Suppose $n < m$. Why is the problem ill-posed in that case? We consider the least-square formulation
$$\min_{p \in \mathbb{R}^n} L(p) := \|A_n^\tau p - y\|_2^2. \tag{2}$$
Show that L is convex, and deduce the normal equations.

4. In this question, we assume that $n = 2$ and $m > n$. Show that the solution of (2) is a line that passes through the arithmetic mean of the points $((t_1, y_1), \ldots, (t_m, y_m))$.

   Hint: With $p = (\alpha, \beta) \in \mathbb{R}^2$, consider the partial derivative of $L(\alpha, \beta)$ with respect to α.

Solution.

1. One checks easily that $\mathcal{A}_n^\tau(\lambda p + \mu q) = \lambda \mathcal{A}_n^\tau(p) + \mu \mathcal{A}_n^\tau(q)$ for all $\lambda, \mu \in \mathbb{R}$, $p, q \in \mathbb{R}_{n-1}[X]$. The representing matrix is
$$A_n^\tau = \begin{pmatrix} 1 & t_1 & \ldots & t_1^{n-1} \\ 1 & t_2 & \ldots & t_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & t_m & \ldots & t_m^{n-1} \end{pmatrix}$$

2. We reason inductively. The formula is easily verifiable for $m = 2$. Let us assume that it holds for a given m. We consider the determinant
$$D_m(X) = \begin{vmatrix} 1 & X & \ldots & X^m \\ 1 & t_1 & \ldots & t_1^m \\ \vdots & \vdots & & \vdots \\ 1 & t_m & \ldots & t_m^m \end{vmatrix}$$
This is a polynomial of degree m, and one has $D(t_i) = 0$ for all $i = 1, \ldots, m$, hence necessarily
$$D_m(X) = a_m \prod_{i=1}^m (X - t_i).$$
The leading coefficient a_m is obtained by developing the determinant with respect to X^m, which gives $a_m = \det(A_m^\tau)$. By the induction hypothesis, we deduce
$$D_m(X) = \prod_{1 \leqslant i < j \leqslant m} (t_j - t_i) \prod_{i=1}^m (X - t_i)$$
and therefore
$$\det(A_{m+1}^\tau) = D_m(t_{m+1}) = \prod_{1 \leqslant i < j \leqslant m+1} (t_j - t_i),$$
which concludes the induction. The linear system admits a unique solution if and only if A_m^τ is injective, hence invertible in the case $m = n$, which happens when $t_i \neq t_j$ for all $i \neq j$ (since then the determinant is nonzero).

3. If $n < m$, then rank $A_m^\tau < m$ and A_m^τ cannot be surjective, so the linear system might have no solution.

   The convexity of L is a simple verification. We can differentiate L with respect to p to obtain
$$\nabla L(p) = 2(A_n^\tau)^\top (A_n^\tau p - y).$$
   The first order optimality conition $\nabla L(p) = 0$ leads to the normal equations.

4. In this situation the problem boils down to finding a line $y(t) = p_0 + p_1 t$ (a polynomial of degree 1) going through the point $(t_1, y_1), \dots, (t_m, y_m)$. The least-squares problem associated with the linear system
$$\begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$
   leads to the normal equations
$$\begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum y_i t_i \end{bmatrix}$$
   which solves in
$$p_2 = \frac{\sum y_i t_i - m\overline{y}\overline{t}}{\sum t_i^2 - m\overline{t}^2}, \quad p_1 = \overline{y} - \beta\overline{t},$$
   where $\overline{y} = m^{-1} \sum y_i$ and $\overline{t} = m^{-1} \sum t_i$. In particular, we do have $\overline{y} = p_1 + p_2\overline{t}$.

■

Exercise 3. Let $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ with $y_1 \leqslant y_2 \leqslant y_3$. We consider the linear system $Ax = y$ for $x \in \mathbb{R}$.

1. Is this system well-posed? why?
2. Let $p \in [1, +\infty]$. We replace the system by the following problem
$$\min_{x \in \mathbb{R}} \|Ax - y\|_p^p \tag{3}$$
   Compute the solution of (3) for $p = 1, 2, \infty$.

Solution.

1. The system is ill-posed, because A is not surjective, since $\operatorname{Ran} A = \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} \; ; \; x \in \mathbb{R} \right\} \neq \mathbb{R}^3$.
2. For $p = 2$, we retrieve the usual least-squares problem, whose solution is given by the normal equations
$$3x = \sum y_i, \quad \textit{i.e.}\ \underline{x = \frac{1}{3}(y_1 + y_2 + y_3)}.$$

For $p = 1$, we want to minimize $\sum |x - y_i|$ over $x \in \mathbb{R}$. For any $x \in \mathbb{R}$, we have that $\sum |x-y_i| \geqslant |x-y_1| + |x-y_3| \geqslant |y_1 - y_3|$, with equality if $x = y_2$ (since $y_1 \leqslant y_2 \leqslant y_3$). Hence the minimum is attained for $\underline{x = y_2}$.

For $p = \infty$, we want to minimize $\max_i |x - y_i|$ for $x \in \mathbb{R}$. Because of the ordering $y_1 \leqslant y_2 \leqslant y_3$, we can see that the maximum over y_1, y_2, y_3 is actually equal to the maximum over y_1 and y_3 only. The point that minimizes the largest distance to one of these two points is their mean: the solution is $\underline{x = \frac{1}{2}(y_1 + y_3)}$.

■

Exercise 4 (An example in infinite dimension). Let $E = \mathrm{L}^2([0,1])$, endowed with the L^2-norm, and let $\mathcal{A}$ be the operator defined by

$$\mathcal{A}f(x) = \int_0^x f(t)\mathrm{d}t$$

1. Check that $A \in \mathcal{L}(E, E)$, and that it is continuous.
2. Show that A is injective.
3. Let $F := \{g \in C^1([0,1]) \ ; \ g(0) = 0\}$. Show that $F \subset \operatorname{Ran} A$. This allows to consider the restriction $\mathcal{A}^{-1}|_F : F \to E$ of $\mathcal{A}^{-1} : \operatorname{Ran} A \to E$.
4. Show that $\mathcal{A}^{-1}|_F$ is not continuous.

   Hint: consider the function $f_n(x) = f(x) + \frac{1}{n}\sin(n^2 x)$ for $f \in C^1([0,1])$ with $f(0) = 0$.

Solution.

1. Linearity is easy to check. One has that $\|\mathcal{A}f\|_2^2 \leqslant \|f\|_2^2$, hence $\mathcal{A}f \in \mathrm{L}^2$. For continuity, note that for any ε, if $\|f - g\|_2 \leqslant \varepsilon$ then

$$\begin{aligned}\|\mathcal{A}f - \mathcal{A}g\|_2^2 &= \int_0^1 \left| \int_0^x (f-g)(t)\mathrm{d}t \right|^2 \mathrm{d}x \\ &\leqslant \int_0^1 \int_0^x |f-g|^2(t)\mathrm{d}t\mathrm{d}x \\ &\leqslant \|f-g\|_2^2 \leqslant \varepsilon^2\end{aligned}$$

2. Let $f \in \operatorname{Ker} \mathcal{A}$. Then, for any x,

$$\int_0^x f(t)\mathrm{d}t = 0,$$

   hence, by derivating,

$$\forall x \in [0,1], \quad f(x) = 0.$$

3. If $g \in F$, then it has a derivative g' and we have that

$$\mathcal{A}g'(x) = \int_0^x g'(t)\mathrm{d}t = g(x) - g(0) = g(x)$$

   hence $g \in \operatorname{Ran} A$.

4. $\mathcal{A}^{-1}|_F$ is simply the standard derivation of a function. We have that

$$\|f_n - f\|_2^2 = \frac{1}{n^2}\int \sin^2(n^2 x)\mathrm{d}x \leqslant \frac{1}{n^2}$$

but on the other hand

$$\|\mathcal{A}^{-1}f_n - \mathcal{A}^{-1}f\|_2^2 = \|f_n' - f'\|_2^2 = n\int_0^1 \cos^2(n^2 x)\mathrm{d}x = \frac{n}{2}\int (1+\cos(2n^2 x))\mathrm{d}x = \frac{n}{2} + \frac{\sin(2n^2)}{4n}.$$

Therefore, $\lim_{n\to\infty}\|f_n - f\| = 0$, but $\lim_{n\to\infty}\|f_n' - f'\| = +\infty$, which shows that $\mathcal{A}^{-1}$ is not continuous.

■